Journal of Statistical Physics, Vol. 108, Nos. 5/6, September 2002 (© 2002)

Gibbsian Dynamics and Invariant Measures for Stochastic Dissipative PDEs

Weinan E[1] **and Di Liu**[2]

Received June 30, 2001; accepted September 29, 2001

We present a general strategy for proving ergodicity for stochastically forced nonlinear dissipative PDEs. It consists of two main steps. The first step is the reduction to a finite dimensional Gibbsian dynamics of the low modes. The second step is to prove the equivalence between measures induced by different past histories using Girsanov theorem. As applications, we prove ergodicity for Ginzburg–Landau, Kuramoto–Sivashinsky and Cahn–Hilliard equations with stochastic forcing.

KEY WORDS: Ergodicity; invariant measures; stationary processes; infinite-dimensional random dynamical systems; stochastic partial differential equations.

1. INTRODUCTION

The main objective of this paper is to prove uniqueness of invariant measures for stochastically forced dissipative PDEs of the form:

$$\frac{\partial u}{\partial t} = -Au + R(u) + \frac{\partial W(x,t)}{\partial t}, \tag{1}$$

when all determining modes are forced. After establishing a general framework to address this question, we present applications to three popular dissipative PDEs: the Ginzburg–Landau equation, the Kuramoto–Sivashinsky equation and the Cahn–Hilliard equation.

Dedicated to David Ruelle and Yasha Sinai on occasion of their sixty-fifth birthdays.

[1] Department of Mathematics and Program in Applied and Computational Mathematics, Princeton University, Princeton, New Jersey 08544; and School of Mathematical Sciences, Peking University, People's Republic of China; e-mail: weinan@princeton.edu

[2] Program in Applied and Computational Mathematics, Princeton University, Princeton, New Jersey 08544; e-mail: dliu@princeton.edu

Technically the main challenge in this program is to prove uniqueness of invariant measures and hence ergodicity for stochastic PDEs with physically realistic stochastic forcing. We still have not achieved this goal. However, progress has been made due to the work of a number of people. Flandoli and Maslowski [FM95] proved uniqueness of the invariant measure for stochastically forced Navier–Stokes equation when the forcing amplitudes on the modes decay algebraically with some rate. In [BKL] and [EMS], uniqueness of the invariant measure for the stochastic Navier–Stokes equation is proved when all determining modes are forced. In [EMatt], E and Mattingly proved uniqueness of the invariant measures for finite-dimensional truncations of the Navier–Stokes equations when only a few (viscosity-independent) large scale modes are forced. Related results for the stochastic Ginzburg–Landau equation and stochastic Navier–Stokes equation can also be found in [EH00], [KS1], and [MY].

Our strategy follows closely that of [EMS] and consists of two steps. The first is to reduce the infinite dimensional Markovian dynamics to the finite dimensional Gibbsian dynamics of the low modes with history dependence. For this finite dimensional Gibbsian dynamics, the noise is non-degenerate, i.e., all modes are forced. The second step is to prove that the measures induced by the dynamics with different past histories are equivalent. This is done by using Girsanov theorem. The main technique here is to truncate the growth of the nonlinear terms so that Girsanov theorem can be appropriately used. This truncation procedure is reminiscent of the standard truncation and mollification procedures in studying distributional solutions of linear PDEs. It is technical in nature, but it does seem to be the main technical obstacle in our work.

As applications, we study three one-dimensional dissipative evolutionary PDEs with periodic boundary condition on $[-\pi, \pi]$:

Stochastic Ginzburg–Landau equation (SGL)

$$\frac{\partial u}{\partial t} = \Delta u + u - u^3 + \frac{\partial W(\cdot, t)}{\partial t}; \tag{2}$$

Stochastic Kuramoto–Sivashinsky equation (SKS)

$$\frac{\partial u}{\partial t} = -\Delta^2 u - \Delta u - u\,\nabla u + \frac{\partial W(\cdot, t)}{\partial t}; \tag{3}$$

Stochastic Cahn–Hilliard equation (SCH)

$$\frac{\partial u}{\partial t} = -\Delta^2 u + \Delta V'(u) + \frac{\partial W(\cdot, t)}{\partial t}. \tag{4}$$

We assume $W(\cdot, t)$ to be of the form

$$W(x, t) = \sum \sigma_k \omega_k(t)\, e_k(x), \tag{5}$$

where the ω_k's are independent standard Wiener processes and $\sigma_k \in \mathbb{R}$. $\{e_k(x), k \in \mathbb{N}\} = \{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \ldots\}$ is the basis of $\mathbb{L}^2[-\pi, \pi]$. Let $[x]$ denote the biggest integer less than or equal to x and define $\mathbb{H}^\alpha = \{u = \sum_{k \in \mathbb{N}} u_k e_k(x), \sum_k [\frac{k}{2}]^{2\alpha} |u_k|^2 < \infty\}$. We will work on the probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ generated by $\{\omega_k\}$. Expectation $\mathbb{E}$ will be taken with respect to $\mathbb{P}$.

For simplicity of presentation, we only consider the case when only the low modes are forced. However, we emphasize that our argument applies with little change to the case when the high modes are also subject to random forcing, as long as the forcing amplitudes decay sufficiently fast. The same comment applies to the results in [EMS].

2. THEORY FOR GENERAL STOCHASTIC DISSIPATIVE PDES

Consider stochastically forced PDEs of the form:

$$du(t) = -Au\, dt + R(u)\, dt + dW(t), \qquad t \geqslant 0, \quad u(0) = u_0, \tag{6}$$

in a separable Hilbert space $\mathbb{H}$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}} \in \mathbb{R}$. A is a self adjoint linear operator on domain $D(A) \subset \mathbb{H}$ with eigenvalues $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N \leqslant \cdots$, $\lim_{k \to \infty} \lambda_k = \infty$ and a complete orthonormal system of eigenvectors $e_1, \ldots, e_N, \ldots$, such that $Ae_i = \lambda_i e_i$. R is a nonlinear function from $D(R) \subset \mathbb{H}$ to $\mathbb{H}$. And

$$W(t) = \sum_{|k| \leqslant N} \sigma_k \omega_k(t)\, e_k(x),$$

where $\{\omega_k\}$'s are independent standard Wiener processes defined on a probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ and $\sigma_k \in \mathbb{R}$, $|\sigma_k| > 0$.

We will assume that Eq. (6) is uniquely solvable for almost all $\omega \in \Omega$ and defines a continuous Markovian semi-group denoted by

$$\varphi^\omega_{s,t} u_0 = u(s, t; \omega, u_0). \tag{7}$$

We simply write φ^ω_t when $s = 0$.

A probability measure μ on $\mathbb{H}$ equipped with the Borel σ-algebra is said to be invariant iff

$$\int_{\mathbb{H}} F(u)\,\mu(du) = \int_{\mathbb{H}} \mathbb{E}F(\varphi_t^\omega u)\,\mu(du) \tag{8}$$

for all bounded continuous functions F on $\mathbb{H}$ and $t \geqslant 0$.

An invariant measure μ can be extended to a measure μ_p on the path space $C((-\infty, 0], \mathbb{H})$. First, define a cylinder set A:

$$A = \{u(s) \in C((-\infty, 0], \mathbb{H}),\, u(t_i) \in A_i,\, i = 0,\dots n\},$$

where $t_0 < t_1 < t_2 \cdots t_n \leqslant 0$ and the A_i's are Borel sets of $\mathbb{H}$. Let $B \subset \mathbb{H} \times \Omega$ to be

$$B = \{(u, \omega),\, u \in A_0,\, \varphi^\omega_{t_0, t_i} u \in A_i,\, i = 1,\dots n\}$$

and define $\mu_p(A) = (\mu \times \mathbb{P})(B)$. Then μ_p is consistent on cylinder sets and can be extended to the natural σ-algebra by Kolmogorov extension theorem.

We define ψ_t^ω to be the map from $C((-\infty, 0], \mathbb{H})$ to $C((-\infty, t], \mathbb{H})$ such that given $u(\cdot) \in C((-\infty, 0], \mathbb{H})$, ψ_t^ω flows forward $u(\cdot)$ with φ from time 0 to t. In other words, $(\psi_t^\omega u)(s) = \varphi_s^\omega u(0)$ for $s \in [0, t]$ and $(\psi_t^\omega u)(s) = u(s)$ for $s \leqslant 0$. Let θ_t be the shift operator such that $(\theta_t v)(s) = v(s+t)$, then $\theta_t \psi_t^\omega$ defines a function from $C((-\infty, 0], \mathbb{H})$ to itself.

If μ is invariant then μ_p is invariant in the sense that

$$\int_{C((-\infty,0],\mathbb{H})} F(u)\, d\mu_p(u) = \mathbb{E}\int_{C((-\infty,0],\mathbb{H})} F(\theta_t \psi_t^\omega u)\, d\mu_p(u) \tag{9}$$

for all bounded functionals F on $C((-\infty, 0], \mathbb{H})$ and $t \geqslant 0$.

Let μ and ν be two invariant measures on $\mathbb{H}$ and let μ_p and ν_p be their respective extensions on $C((-\infty, 0], \mathbb{H})$, it is obvious that $\mu_p = \nu_p$ implies $\mu = \nu$.

2.1. Gibbsian Dynamics

In this section, we will introduce the notion of Gibbsian dynamics of the low modes. We partition $\mathbb{H}$ into two subspaces $\mathbb{H} = \mathbb{H}_\ell \oplus \mathbb{H}_h$ defined as:

$$\mathbb{H}_\ell = \operatorname{span}\{e_k,\, |k| \leqslant N\}, \qquad \mathbb{H}_h = \operatorname{span}\{e_k,\, |k| > N\}.$$

We will call $\mathbb{H}_\ell$ the space of low modes and $\mathbb{H}_h$ the space of high modes. Denote by P_ℓ and P_h the projections onto $\mathbb{H}_\ell$ and $\mathbb{H}_h$, respectively. Let

$\ell = P_\ell u$ and $h = P_h u$. We write $u(t) = (\ell(t), h(t))$ and rewrite the stochastic equation (6) in terms of $\ell(t)$ and $h(t)$:

$$d\ell(t) = [-A\ell + P_\ell R(u)]\,dt + dW(t), \tag{10}$$

$$\frac{dh(t)}{dt} = -Ah + P_h R(u). \tag{11}$$

We will show that for statistically invariant solutions of (6) existing for time from $-\infty$ to $+\infty$, h is uniquely determined by the past history of ℓ from $-\infty$ to 0 for almost all $u(t)$.

We will impose a number of conditions on (6).

Condition 1. There exist constants $\eta > 0$ and $k_0 \geqslant 0$ such that

$$-\langle Ax, x\rangle_{\mathbb{H}} + \langle R(x), x\rangle_{\mathbb{H}} \leqslant -\eta\, |x|_{\mathbb{H}}^2 + k_0. \tag{12}$$

Condition 1 guarantees that basic energy estimates hold for (6). Define $\mathscr{E}_0 = \sum |\sigma_k|^2$. The following lemma will be proved in Section 4.1.

Lemma 2.1. Let μ be an invariant measure on $\mathbb{H}$ and let μ_p be the corresponding measure induced on $C((-\infty, 0], \mathbb{H})$. Then under Condition 1, $\forall K_0 > 0$ and $\delta > \frac{1}{2}$, for μ_p-almost every trajectory $u(\cdot)$ in $C((-\infty, 0], \mathbb{H})$, $\exists T_1$ such that for $s \leqslant 0$

$$|u(s)|_{\mathbb{H}}^2 \leqslant 2k_0 + \mathscr{E}_0 + K_0 \max(T_1, |s|)^\delta. \tag{13}$$

Condition 2. Let $u_1, u_2 \in \mathbb{H}$ and let $\rho = u_1 - u_2$. There exist a constant $\alpha \in [0, 1)$ and a non-negative function $K(u)$ on $\mathbb{H}$ such that

$$\langle R(u_1) - R(u_2), \rho\rangle_{\mathbb{H}} \leqslant \alpha \langle A\rho, \rho\rangle_{\mathbb{H}} + K(u_1)\, |\rho|_{\mathbb{H}}^2. \tag{14}$$

Furthermore,

$$\int_{\mathbb{H}} K(u)\, d\mu(u) \leqslant \beta \tag{15}$$

for some constant β independent of the invariant measure μ.

A consequence of the second part of Condition 2 is that given any ergodic invariant measure μ, for μ_p-almost all $u(\cdot) \in C((-\infty, 0], \mathbb{H})$:

$$\lim_{t_0 \to -\infty} \frac{1}{t - t_0} \int_{t_0}^{t} K(u(s))\, ds \leqslant \beta. \tag{16}$$

Define the set $U \subset C((-\infty, 0], \mathbb{H})$ to consist of all v: $((-\infty, 0] \to \mathbb{H})$ such that v satisfies (13) and (16). By definitions, if Conditions 1 and 2 are satisfied, then for any ergodic invariant measure μ, $\mu_p(U) = 1$.

We will use $\ell(t)$ to refer to the value of the low mode at time t and will use L^t to mean the entire trajectory from $-\infty$ to t. Hence $\ell(t) \in \mathbb{H}_\ell$ and $L^t \in C((-\infty, t], \mathbb{H}_\ell)$ and $\ell(s) = L^t(s)$ for $0 \leqslant s \leqslant t$. By $\Phi_s(L^t, h_0)$ with $s \leqslant t$, we mean the solution to (11), the equation for the high mode, at time s with initial condition h_0 and low mode forcing L^t. Of course $\Phi_s(L^t, h_0)$ only depends on the information of L^t between 0 and s. So is $\Phi_{t_0, t}(L^t, h_0)$ defined for the solutions starting from t_0.

Lemma 2.2. Under Conditions 1 and 2, if we choose N sufficiently large such that

$$-\gamma = -(1-\alpha)\,\lambda_N + \beta < 0, \tag{17}$$

then the following holds for any ergodic invariant measure μ:

If there exist two solutions of the form $u_1(t) = (\ell(t), h_1(t))$, $u_2(t) = (\ell(t), h_2(t)) \in U$, then $u_1 = u_2$, i.e., $h_1 = h_2$.

Moreover if $u(t) = (\ell(t), h(t)) \in U$ is a solution, then for any $h_0 \in \mathbb{H}_h$ and $t \leqslant 0$, we have

$$\lim_{t_0 \to -\infty} \Phi_{t_0, t}(L^t, h_0) = h(t).$$

Proof of Lemma 2.2. Let $\rho(t) = h_1(t) - h_2(t)$. From (11) we have

$$\frac{d\rho}{dt} = -A\rho + P_h[R(u_1) - R(u_2)].$$

Taking inner product with ρ and by Condition 2, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\rho|_{\mathbb{H}}^2 &= -\langle A\rho, \rho\rangle_{\mathbb{H}} + \langle R(u_1) - R(u_2), \rho\rangle_{\mathbb{H}} \\ &\leqslant -(1-\alpha)\,\lambda_N\,|\rho|_{\mathbb{H}}^2 + K(u_1)\,|\rho|_{\mathbb{H}}^2.\end{aligned}$$

By the definition of U, $\exists T_2$ depending on t and u_1 such that for $t_0 < T_2$,

$$-(1-\alpha)\,\lambda_N(t-t_0) + \int_{t_0}^{t} K(u_1(s))\,ds \leqslant -\frac{\gamma}{2}(t-t_0).$$

Hence we have, for $t_0 < T_2$,

$$|\rho(t)|_{\mathbb{H}}^2 \leqslant |\rho(t_0)|_{\mathbb{H}}^2 \exp\left\{-2(1-\alpha)\,\lambda_N(t-t_0)+2\int_{t_0}^t K(u_1(s))\,ds\right\}$$
$$\leqslant |\rho(t_0)|_{\mathbb{H}}^2 \exp\{-\gamma(t-t_0)\}.$$

By Lemma 2.1 we have for any $t_0 \leqslant \min\{T_1, T_2\}$,

$$|\rho(t)|_{\mathbb{H}}^2 \leqslant 2[2k_0+\mathscr{E}_0+|t_0|^{\frac{2}{3}}]\exp\{-\gamma(t-t_0)\} \to 0,$$

as $t_0 \to -\infty$. This completes the proof of the first part of Lemma 2.2.

For the second part, let the high mode of the given solution $u(t)$ be h_1 and the solution to (11) starting from t_0 and h_0 be h_2, then we have the estimate

$$|\rho(t)|_{\mathbb{H}}^2 \leqslant |(h(t_0)-h_0)|_{\mathbb{H}}^2 \exp\left(-2(1-\alpha)\,\lambda_N(t-t_0)+2\int_{t_0}^t K(u(s))\,ds\right).$$

By the same argument, $\rho(t)$ goes to zero as $t_0 \to -\infty$. Hence the limit exists and equals $h(t)$. ∎

From now on, we assume that N is large enough such that (17) holds.

Denote by $\mathscr{P}$ the set of all $\ell(\cdot) \in C((-\infty, 0], \mathbb{H}_\ell)$ such that $\ell = P_\ell u$ for some $u=(\ell, h) \in U$. By the assumption for the existence of the solution, we know the set $\mathscr{P}$ is not empty. Because of Lemma 2.2, we can define the map Φ_0 which reconstructs the high modes of the solution at time zero from given low mode trajectories stretching from zero back to $-\infty$. In this notation $h(0)=\Phi_0(L^0)$ where L^0 is some "low mode past" in $\mathscr{P}$.

Define $\Phi_t(L^t)=\Phi_t(L^t, \Phi_0(L^0))$. Now given any initial low mode past of $L^0 \in \mathscr{P}$, we can solve the future of ℓ using the Gibbsian dynamics:

$$d\ell(t) = [-A\ell + G(\ell(t), \Phi_t(L^t))]\,dt + dW(t), \tag{18}$$

where

$$G(\ell, h) = P_\ell R(\ell+h).$$

Thus we have a closed form for the dynamics of the low modes given an initial past $L^0 \in \mathscr{P}$. We write $L^t = \mathsf{S}_t^\omega L^0$.

2.2. Equivalence Between Measures

In this section, we prove that the measures induced by the Gibbsian dynamics with different past histories are equivalent. This is done using

Girsanov theorem. Some conditions are necessary to ensure that the nonlinear terms do not grow too fast so that Girsanov's theorem can be safely used.

Given any $L^0 \in \mathscr{P}$, let $Q_t(L^0, \cdot)$ be the measure induced on $C([0, t], \mathbb{H}_\ell)$ by the dynamics of the equation starting from L^0. In other words, $Q_t(L^0, \cdot)$ is the distribution of $\mathsf{S}_t^\omega L^0$ viewed as a random variable taking values in $C([0, t], \mathbb{H}_\ell)$. Similarly let $Q_\infty(L^0, \cdot)$ be the distribution induced on $C([0, \infty), \mathbb{H}_\ell)$ starting from L^0. We also denote by $R_t(L^0, \cdot)$ the distribution of $\ell(t)$ on $\mathbb{H}_\ell$ conditioned at starting from L^0 at time zero.

Define $D(g, f_1, f_2) \stackrel{\text{def}}{=} G(g, f_1) - G(g, f_2)$. Suppose $L^t = \mathsf{S}_t^\omega L^0$ for some $L^0 \in \mathscr{P}$ and $\bar{h}_0$ be some high mode initial condition in $\mathbb{H}_h$. Let $h(t) = \Phi_t(L^t, \Phi_0(L^0)) = \Phi_t(L^t)$ and $\bar{h}(t) = \Phi_t(L^t, \bar{h}_0)$. It should be mentioned that $(\ell(t), h(t))$ constitutes a solution for the stochastic equation (6) while $(\ell(t), \bar{h}(t))$ is not necessarily a solution.

Now we impose two conditions on the Gibbsian dynamics (18).

Condition 3. $\forall L^0 \in \mathscr{P}$, $\bar{h}_0 \in \mathbb{H}_h$ and $a \in (0, 1)$, $\exists K > 0$ such that

$$\mathbb{P}\left\{\int_0^\infty |D(\ell(t), h(t), \bar{h}(t))|_{\mathbb{H}}^2 \, dt < K\right\} > 1 - a > 0. \tag{19}$$

Condition 4. $\forall L^0 \in \mathscr{P}$, $a \in (0, 1)$ and $T > 0$, $\exists K > 0$ such that

$$\mathbb{P}\left\{\int_0^T |G(\ell(s), h(t))|_{\mathbb{H}}^2 \, ds < K\right\} > 1 - a > 0. \tag{20}$$

Lemma 2.3. Assume that Condition 3 holds. Let L_0^1 and L_0^2 be two initial pasts in $\mathscr{P}$ such that $L_1^0(0) = L_2^0(0)$, then $Q_\infty(L_1^0, \cdot)$ and $Q_\infty(L_2^0, \cdot)$ are mutually equivalent.

Lemma 2.4. Under Condition 4, $\forall L^0 \in \mathscr{P}$, $R_t(L^0, \cdot)$ is equivalent to the Lebesgue measure $m(\cdot)$.

For any measure μ on $\mathbb{H}$, let $P_\ell \mu$ be its projection to the low modes space $\mathbb{H}_\ell$. Namely, $(P_\ell \mu)(B) = \mu(P_\ell^{-1}(B))$. Then we have the following direct consequence of Lemma 2.4.

Corollary 2.5. Under Condition 4, if μ is an ergodic invariant measure then $P_\ell \mu$ has a component which is equivalent to the Lebesgue measure.

Proof of Lemma 2.3. Define

$$A(K) = \Big\{ F^t \in C([0,\infty), \mathbb{H}_\ell) :$$

$$\int_0^\infty |D(F^t(s), \Phi_s(F^s, h_1(0)), \Phi_s(F^s, h_2(0)))|_{\mathbb{H}}^2 \, dt < K \Big\},$$

where $h_i(0) = \Phi_0(L_i^0)$, $i = 1, 2$.

Then Condition 3 says that we can choose K big enough such that

$$\mathbb{P}\{\omega: S_t^\omega L_i^0 \in A(K)\} > 1 - a, \qquad i = 1, 2.$$

Hence, $Q_\infty(L_i^0, A(K)) > 1 - a$. Since a is arbitrary, it is sufficient to prove that for any choice of $K > 0$, $Q_\infty(L_1^0, \cdot \cap A(K))$ is equivalent to $Q_\infty(L_2^0, \cdot \cap A(K))$.

We consider the following truncated processes y which will agree with ℓ on the set $A = A(K)$. As before, $y(t)$ denotes the value of the process at time t and Y^t means the entire trajectory up to time t.

$$dy_i(t) = [-Ay_i(t) + \Theta_t(Y_i^t)\, G(y_i(t), \Phi_t(Y_i^t, h_i(0)))]\, dt + dW(t),$$

$$y_i(0) = \ell_i(0),$$

where

$$\Theta_t(f) = \begin{cases} 1 & \text{if } f \in A(K)|_{[0,t]}, \\ 0 & \text{if } f \notin A(K)|_{[0,t]}. \end{cases}$$

$A(K)|_{[0,T]}$ is the set of the low mode paths which stay in $A(K)$ up to time T.

Let $Q_\infty^y(L_1^0, \cdot)$ and $Q_\infty^y(L_2^0, \cdot)$ be the measures induced by Y_1 and Y_2 respectively. Girsanov theorem will imply the result if the corresponding Novikov condition holds:

$$\mathbb{E} \exp \Big\{ \tfrac{1}{2} \int_0^\infty |\Sigma^{-1} \Theta_t(Y_1^t)\, D(y_1(t), \Phi_t(Y_1^t, h_1(0)), \Phi_t(Y_1^t, h_2(0)))|_{\mathbb{H}}^2 \, dt \Big\} < \infty,$$

where Σ is a diagonal matrix with the σ_k's on the diagonal, i.e., $\Sigma = \mathrm{diag}(\sigma_1, ..., \sigma_N)$. Since $|\Sigma^{-1}| < \infty$, it would be enough to show that

$$\sup_\omega \int_0^\infty |\Theta_t(Y_1^t)\, D(y_1(t), \Phi_t(Y_1^t, h_1(0)), \Phi_t(Y_1^t, h_2(0)))|_{\mathbb{H}}^2 \, dt < K < \infty,$$

which is implied by the definitions of $A(K)$ and Θ and the fact that y agrees with ℓ on $A(K)$. ▌

Proof of Lemma 2.4. Fix $L^0 \in \mathscr{P}$. The proof proceeds by comparing the process $\ell(t)$ to the process $x(t)$ defined by the following stochastic ODE:

$$dx(t) = -Ax(t)\,dt + dW(t), \qquad x(0) = \ell(0).$$

And define A_T to be

$$A_T(b_0) = \left\{ F^t \in C([0, \infty), \mathbb{H}_\ell) : \int_0^T |G(F^t(s), \Phi_s(F^s, h_0))|_{\mathbb{H}}^2\, ds < b_0 \right\},$$

where $h_0 = \Phi_0(L^0)$ and b_0 is an arbitrary positive constant.

We use the truncation technique again. Define $z(t)$ to be the solution of:

$$dz(t) = [-Az(t) + \Theta_t(Z^t)\, G(z(t), \Phi_t(Z^t, h_0))]\, dt + dW(t), \qquad z(0) = \ell(0).$$

As above, $\Theta_t(Z^t)$ is a cut-off function defined as:

$$\Theta_t(f) = \begin{cases} 1 & \text{if } \ f \in A_T|_{[0,t]}, \\ 0 & \text{if } \ f \notin A_T|_{[0,t]}. \end{cases}$$

Let $Q_t^x(L^0, \cdot)$ and $Q_t^\ell(L^0, \cdot)$ be the two measures induced on $C([0, t], \mathbb{H}_\ell)$ by the dynamics of x and ℓ respectively. Observe that $z(t) = l(t)$ as long as the trajectories stay in A_T, the Girsanov theorem will imply $Q_t^x(L^0, A_T)$ is equivalent to $Q_t^\ell(L^0, A_T)$ for $0 \leqslant t \leqslant T$ if the following Novikov condition holds:

$$\mathbb{E} \exp\left\{ \frac{1}{2} \int_0^t |\Sigma^{-1}\Theta_s(Z^s)|^2\, |G(z(s), \Phi_s(Z^s, h_0))|_{\mathbb{H}}^2\, ds \right\} < \infty.$$

It is sufficient to prove the stronger condition

$$\sup_{z(\cdot) \in A_T} \int_0^t |G(z(s), \Phi_s(Z^s, h_0))|_{\mathbb{H}}^2\, ds < \infty,$$

which is implied by the definition of A_T.

By Condition 4, we can make the measure of A_T as close as enough to 1 by increasing b_0. Then we can conclude that $Q_t^x(L^0, \cdot)$ is equivalent to

$Q_t^\ell(L^0, \cdot)$. Notice that $x(t)$ is an Ornstein–Unlenbeck process with non-degenerate noise, and thus a Gaussian random variable with positive density. Its distribution is equivalent to the Lebesgue measure. So we know that $R_t(L^0, \cdot)$ is equivalent to the Lebesgue measure. ∎

2.3. Uniqueness of the Invariant Measure

Let μ be an ergodic invariant measure on $\mathbb{H}$ for dynamics (6) and μ_p be its extensions to the path space $C((-\infty, 0], \mathbb{H})$. We will also consider the restriction of μ_p to $C((-\infty, 0], \mathbb{H}_\ell)$, still denoted by μ_p. Consider the stochastic process defined by $\theta_t \mathsf{S}_t^\omega L^0$ where L^0 is a random variable on $\mathscr{P}$ distributed according to an invariant measure μ_p. For $t \geqslant 0$ it is a random process with values in $\mathscr{P}$. Since μ_p is invariant with respect to the dynamics, $\theta_t \mathsf{S}_t^\omega L^0$ is a stationary random process. Hence with probability one there exist time averages along trajectories $\theta_t \mathsf{S}_t^\omega L^0$.

Take any bounded measurable functional F from $C((-\infty, 0], \mathbb{H}_\ell) \to \mathbb{R}$ such that $F(L^0)$ depends only on L^0 on a finite time interval. Let

$$\bar{F} = \int F(L)\, d\mu_p(L). \tag{21}$$

Theorem 1. Suppose that the stochastic PDE (6) satisfies Conditions 1–4 and N is large enough such that (17) holds , then (6) has a unique invariant measure.

The proof here is basically the same as the one given in [EMS] for the stochastic Navier–Stokes equation. We give it here for self-completion.

Proof of Theorem 1. Suppose μ_1 and μ_2 are two different ergodic invariant measures on $\mathbb{H}$. Then they are mutually singular. Let $\mu_{p,1}$ and $\mu_{p,2}$ be their extensions onto the path space $\mathscr{P}$, we can find a functional F defined as above such that $\bar{F}_1 = \int F(L)\, d\mu_{p,1}(L) \neq \bar{F}_2 = \int F(L)\, d\mu_{p,2}(L)$. Let L_i^0 be a random variable on $\mathscr{P}$ distributed as $\mu_{p,i}$. The limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T F(\theta_t \mathsf{S}_t^\omega L_i^o)\, dt = \bar{F}_i$$

is well defined for $\mathbb{P}$-almost every ω.

For $\ell \in \mathbb{H}_\ell$, define $\mathscr{P}^i(\ell) = \{L \in \mathscr{P} : L(0) = \ell\}$ and let $\mu_{p,i}(\cdot \mid \ell)$ be the conditional measure that $L(0) = \ell$. By Fubini's theorem, we know that for $P_\ell \mu_i$-almost every $\ell \in \mathbb{H}_\ell$ we have $\mu_{p,i}(\mathscr{P}^i(\ell) \mid \ell) = 1$. Hence we can find a set $A_i \subset \mathbb{H}_\ell$ such that $\mu_{p,i}(\mathscr{P}^i(\ell) \mid \ell) = 1$ for all $\ell \in A_i$ and $P_\ell \mu_i(A_i) = 1$.

Define $A = A_1 \cap A_2$. Corollary 2.5 implies that $P_\ell \mu_i(A) > 0$ for $i = 1, 2$. Hence there exists some $\ell^* \in A$.

Since $\ell^* \in A_1 \cap A_2$, we know that $\mu_{p,i}(\mathscr{P}^i(\ell^*) \mid \ell^*) = 1$ for $i = 1, 2$. Thus there exist some $L_{*,1} \in \mathscr{P}^1(\ell^*)$ and $L_{*,2} \in \mathscr{P}^2(\ell^*)$. Notice that by construction $L_{*,1}(0) = \ell^* = L_{*,2}(0)$ and hence it follows from Lemma 2.3 that $Q_\infty(L_{*,1}, \cdot)$ and $Q_\infty(L_{*,2}, \cdot)$ are equivalent. Since $L_{*,i} \in \mathscr{P}^i(\ell^*)$, we know that we can pick $B_i \subset C([0, \infty), \mathbb{H})$ such the time average of F converges to $\bar{F}_i$ for all futures in B_i and $Q_\infty(L_{*,i}, B_i) = 1$ for $i = 1, 2$. Since the Q's are equivalent, $Q_\infty(L_{*,i}, B_1 \cap B_2) > 0$ and hence $B_1 \cap B_2$ is non-empty. This in turn implies that $\bar{F}_1 = \bar{F}_2$ which contradicts the assumption that they were not equal. ▌

3. APPLICATIONS

In this section, we will discuss three popular stochastic PDEs introduced in the first section. We will show that they satisfy the conditions given in last section for the uniqueness of the invariant measure. Projecting (2), (3) and (4) onto $\mathbb{L}^2$, we obtain the following Itô stochastic systems:

Stochastic Ginzburg–Landau equation (SGL)

$$du(x, t) = (\Delta u + u - u^3)\, dt + dW(x, t); \tag{22}$$

Stochastic Kuramoto–Sivashinsky equation (SKS)

$$du(x, t) = -(\Delta^2 u + \Delta u + u \nabla u)\, dt + dW(x, t); \tag{23}$$

Stochastic Cahn–Hilliard equation (SCH)

$$du(x, t) = (-\Delta^2 u + \Delta V'(u))\, dt + dW(x, t). \tag{24}$$

The existence and uniqueness of the solution for the initial value problem associated with stochastic Ginzburg–Landau equation (22) in $\mathbb{H}^1$ can be found in [DPZ96] as a special case of the dissipative equations. The existence of at least one invariant measure is also given in [DPZ96]. With our assumptions on $V(x)$, the stochastic Cahn–Hilliard equation is also dissipative, so the same results for the SCH equation (24) can be given in the same way. As to the stochastic Kuramoto–Sivashinsky equation (23), notice that the linear part on the right side is the generator of a contractive C_0-semigroup on $\mathbb{H}^2$ and the nonlinear term is of the Burgers type. Hence based on Lemma 3.2 later, the same results for SKS equation (23) can be proved using an argument similar to that for the Burgers equation in [DPZ96].

3.1. Ginzburg–Landau Equation

The notations are those of Section 2. $\mathbb{H}$ is the space $\mathbb{L}^2[-\pi, \pi]$. And for every $v \in \mathbb{H}^2$,

$$Av = -\Delta v, \qquad R(v) = v - v^3.$$

The eigenvectors of A are

$$\{e_k(x), k \in \mathbb{N}\} = \left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} \cos x, \frac{1}{\sqrt{2\pi}} \sin x, \ldots, \frac{1}{\sqrt{2\pi}} \cos nx, \frac{1}{\sqrt{2\pi}} \sin nx, \ldots\right\}$$

with eigenvalues $\lambda_n = [\frac{n}{2}]^2$, $n = 1, 2, \ldots$. Here $[x]$ means the biggest integer less than or equal to x.

The Poincaré inequality gives

$$|\nabla v|^2_{\mathbb{L}^2} + |v|^2_{\mathbb{L}^1} \geqslant |v|^2_{\mathbb{L}^2} \qquad \text{and} \qquad |\triangle v|^2_{\mathbb{L}^2} \geqslant |\nabla v|^2_{\mathbb{L}^2}. \tag{25}$$

And the Sobolev inequality in one dimension has the form:

$$2\, |v|^2_{\mathbb{L}^\infty} \leqslant |v|^2_{\mathbb{L}^2} + |\nabla v|^2_{\mathbb{L}^2}. \tag{26}$$

Since $\mathbb{H}^1 \subset \mathbb{L}^4$ and

$$|u(t)|_{\mathbb{L}^1} \leqslant (2\pi)^{\frac{1}{2}}\, |u(t)|_{\mathbb{L}^2}, \qquad |u(t)|_{\mathbb{L}^2} \leqslant (2\pi)^{\frac{1}{4}}\, |u(t)|_{\mathbb{L}^4}, \tag{27}$$

we have

$$|u(t)|^2_{\mathbb{L}^1} + |u(t)|^2_{\mathbb{L}^2} - |u(t)|^4_{\mathbb{L}^4} \leqslant (2\pi + 1)\, |u(t)|^2_{\mathbb{L}^2} - (2\pi)^{-1}\, |u(t)|^4_{\mathbb{L}^2} \leqslant k_0$$

for some constant k_0 and hence

$$\begin{aligned} -\langle Au, u\rangle_{\mathbb{L}^2} + \langle R(u), u\rangle_{\mathbb{L}^2} &= -|\nabla u|^2_{\mathbb{L}^2} + |u|^2_{\mathbb{L}^2} - |u|^4_{\mathbb{L}^4} \\ &= -|\nabla u|^2_{\mathbb{L}^2} - |u(t)|^2_{\mathbb{L}^1} + |u(t)|^2_{\mathbb{L}^1} + |u|^2_{\mathbb{L}^2} - |u|^4_{\mathbb{L}^4} \\ &\leqslant -|u|^2_{\mathbb{L}^2} + k_0. \end{aligned}$$

This establishes Condition 1 for the SGL equation with $\eta = 1$ and k_0.

Let $\rho = u_1 - u_2$, we have

$$\langle R(u_1 - u_2), \rho\rangle_{\mathbb{L}^2} = |\rho|^2_{\mathbb{L}^2} - \langle (u_1^3 - u_2^3), \rho\rangle_{\mathbb{L}^2} \leqslant |\rho|^2_{\mathbb{L}^2},$$

which means the SGL equation satisfies Condition 2 with $\alpha = 0$, $K(u) = 1$ and $\beta = 1$. (17) is equivalent to $N > 3$. From now on, we assume $N > 3$.

The following lemma describes the the growth rate of $|u(t)|_{\mathbb{L}^2}^2$ and $|\nabla u(t)|_{\mathbb{L}^2}^2$ on a set with probability arbitrarily close to 1.

Lemma 3.1. $\forall \delta > \frac{1}{2}$, $a \in (0, 1)$ and $C_0 > 0$, $\exists C(\delta, a, C_0) > 0$ such that if $|u_0|_{\mathbb{L}^2}^2 + |\nabla u_0|_{\mathbb{L}^2}^2 < C_0$,

$$\mathbb{P}\left\{|u(t)|_{\mathbb{L}^2}^2 + |\nabla u(t)|_{\mathbb{L}^2}^2 + 2\int_0^t |\triangle u(s)|_{\mathbb{L}^2}^2\, ds \leqslant C_0 + C_1 t + C(t+1)^\delta \text{ for all } t \geqslant 0\right\}$$

$$\geqslant 1 - a,$$

where $C_1 = 2k_0 + \mathscr{E}_0 + \mathscr{E}_1$ and $\mathscr{E}_1 = \sum [\frac{k}{2}]^2 |\sigma_k|^2$.

Proof. Applying Itô's formula to the map $u(t) \mapsto |u(t)|_{\mathbb{L}^2}^2$ and $u(t) \mapsto |\nabla u(t)|_{\mathbb{L}^2}^2$ produces

$$d|u(t)|_{\mathbb{L}^2}^2 = 2[-|\nabla u(t)|_{\mathbb{L}^2}^2\, dt + |u(t)|_{\mathbb{L}^2}^2\, dt - |u(t)|_{\mathbb{L}^4}^4\, dt + \langle u(t), dW\rangle_{\mathbb{L}^2}] + \mathscr{E}_0\, dt \tag{28}$$

and

$$\begin{aligned} d|\nabla u(t)|_{\mathbb{L}^2}^2 &= 2[-|\triangle u(t)|_{\mathbb{L}^2}^2\, dt + |\nabla u(t)|_{\mathbb{L}^2}^2\, dt - 3\,|u(t)\,\nabla u(t)|_{\mathbb{L}^2}^2\, dt \\ &\quad - \langle \triangle u(t), dW\rangle_{\mathbb{L}^2}] + \mathscr{E}_1\, dt \\ &\leqslant 2[-|\triangle u(t)|_{\mathbb{L}^2}^2\, dt + |\nabla u(t)|_{\mathbb{L}^2}^2\, dt - \langle \triangle u(t), dW\rangle_{\mathbb{L}^2}] + \mathscr{E}_1\, dt. \end{aligned} \tag{29}$$

Combining with (28) and (29) and using inequality (27) give the energy inequality after integration

$$\begin{aligned} &|u(t)|_{\mathbb{L}^2}^2 + |\nabla u(t)|_{\mathbb{L}^2}^2 + 2\int_0^t |\triangle u(s)|_{\mathbb{L}^2}^2\, ds \\ &\quad \leqslant |u_0|_{\mathbb{L}^2}^2 + |\nabla u_0|_{\mathbb{L}^2}^2 + (2k_0 + \mathscr{E}_0 + \mathscr{E}_1)\, t \\ &\quad \leqslant +2\int_0^t \langle u(s), dW(s)\rangle_{\mathbb{L}^2} - 2\int_0^t \langle u(s), \triangle dW(s)\rangle_{\mathbb{L}^2}. \end{aligned}$$

Let $M_t = \int_0^t \langle u(s), dW(s)\rangle_{\mathbb{L}^2}$ and $M_t^1 = -\int_0^t \langle u(s), \triangle dW(s)\rangle_{\mathbb{L}^2}$. Since $|u_0|_{\mathbb{L}^2}^2 + |\nabla u_0|_{\mathbb{L}^2}^2 < C_0$, we only need to show that for C large enough

$$\mathbb{P}\{2M_t + 2M_t^1 \leqslant C(t+1)^\delta \text{ for } t \geqslant 0\} \geqslant 1 - a.$$

The quadratic variation $[M, M]_t$ and $[M^1, M^1]_t$ satisfy the inequalities

$$[M, M]_t \leqslant \sigma_{\max}^2 \int_0^t |u(s)|_{\mathbb{L}^2}^2, \qquad [M^1, M^1]_t \leqslant (\Delta\sigma)_{\max}^2 \int_0^t |u(s)|_{\mathbb{L}^2}^2,$$

where $\sigma_{\max}^2 = \sup |\sigma_k|^2$ and $(\Delta\sigma_{\max})^2 = \sup |k^2\sigma_k|^2$. Hence

$$([M, M]_t)^p \leqslant \sigma_{\max}^{2p} \left(\int_0^t |u(s)|_{\mathbb{L}^2}^2\right)^p \leqslant \sigma_{\max}^{2p} t^{p-1} \int_0^t |u(s)|_{\mathbb{L}^2}^{2p}\, ds,$$

$$([M^1, M^1]_t)^p \leqslant (\Delta\sigma)_{\max}^{2p} \left(\int_0^t |u(s)|_{\mathbb{L}^2}^2\right)^p \leqslant (\Delta\sigma)_{\max}^{2p}\, t^{p-1} \int_0^t |u(s)|_{\mathbb{L}^2}^{2p}\, ds.$$

From Corollary 4.2, we know that if $|u(0)|_{\mathbb{L}^2}^2 < C_0$ then for any $p \geqslant 1$ there exists a constant C_p so that $\mathbb{E}\,|u(t)|_{\mathbb{L}^2}^{2p} \leqslant C_p$ for all $t \geqslant 0$. Define the events

$$A_k = \left\{ \sup_{s \in [0, k]} |M_s| > \frac{C}{4} k^\delta \right\}.$$

By the Doob–Kolmogorov martingale inequality and Martingale Moment inequality we have

$$\mathbb{P}\{A_k\} \leqslant \frac{4^{2p}\mathbb{E}\,|M_k|^{2p}}{C^{2p}k^{2p\delta}} \leqslant \frac{4^{2p}\acute{C}_p\mathbb{E}([M, M]_k)^p}{C^{2p}k^{2p\delta}} \leqslant \frac{4^{2p}\sigma_{\max}^{2p}\acute{C}_p C_p}{C^{2p}} \frac{k^p}{k^{2p\delta}}.$$

And notice that

$$\mathbb{P}\left\{M_t \leqslant \frac{C}{4}(t+1)^\delta \text{ for } t \geqslant 0\right\} \geqslant 1 - \mathbb{P}\left\{\bigcup_k A_k\right\} \geqslant 1 - \sum_k \mathbb{P}\{A_k\}.$$

For the sum of $\mathbb{P}\{A_k\}$ to be finite, we only need $\delta > \frac{1+(1/p)}{2}$. And the sum can be made as small as we want by increasing C. By a similar argument $\mathbb{P}\{M_t^1 \leqslant \frac{C}{4}(t+1)^\delta$ for all $t \geqslant 0\}$ can also be made as close as enough to 1 by increasing C. Let C to be big enough such that $\mathbb{P}\{M_t > \frac{C}{4}(t+1)^\delta$ for some $t \geqslant 0\} < \frac{a}{2}$ and $\mathbb{P}\{M_t^1 > \frac{C}{4}(t+1)^\delta$ for some $t \geqslant 0\} < \frac{a}{2}$. Then

$$\mathbb{P}\{2M_t + 2M_t^1 \leqslant C(t+1)^\delta \text{ for } t \geqslant 0\} \geqslant 1 - a.$$

By the arbitrariness of p, we have the conclusion. ∎

Next we show that the SGL equation satisfies the Condition 3. Fix $L^0 \in \mathscr{P}$ and $\bar{h}(0)$ a high mode initial value at time zero. Let $L^s = \mathsf{S}_s^\omega L^0$

and $\ell(s)=L^t(s)$ for $s\leqslant t$. Then with probability one, $h(s)=\Phi_s(L^s)$ where $u(s)=(\ell(s),h(s))$. Fix a constant C_0 such that $|u(0)|^2_{\mathbb{L}^2}+|\nabla u(0)|^2_{\mathbb{L}^2}\leqslant C_0$. For any positive C we define

$$D(C)=\Big\{f\in C([0,\infty),\mathbb{L}^2_\ell):$$

$$|v(t)|^2_{\mathbb{L}^2}+|\nabla v(t)|^2_{\mathbb{L}^2}+2\int_0^t|\Delta v(s)|^2_{\mathbb{L}^2}\,ds<C_0+(2k_0+\mathscr{E}_0+\mathscr{E}_1)\,t+Ct^{\frac{4}{5}},$$

$$\text{where } v(s)=f(s)+\Phi_s(f,\Phi_0(L^0))\Big\}.$$

Projecting $u(t)$ onto $\mathbb{H}_\ell$, by Lemma 3.1 we know that for any $a\in(0,1)$, there exists a C such that

$$\mathbb{P}\{\omega:\mathbf{S}^\omega_tL^0\in D(C)\}>1-a>0.$$

Putting $\bar{h}(s)=\Phi_s(L^s,\bar{h}(0))$, $\rho(s)=h(s)-\bar{h}(s)$, then $u=\ell+h=\ell+\bar{h}+\rho$ and we have

$$\begin{aligned}
&|D(\ell(s),h(s),\bar{h}(s))|^2_{\mathbb{L}^2}\\
&\quad=\sup_{w\in\mathbb{L}^2,|w|=1}|\langle P_\ell(\rho[u^2+u(u-\rho)+(u-\rho)^2],w\rangle|^2\\
&\quad\leqslant\sup_{w\in\mathbb{L}^2,|w|=1}(|\nabla P_\ell w|^2_{\mathbb{L}^2}+|P_\ell w|^2_{\mathbb{L}^2})\,|\rho[u^2+u(u-\rho)+(u-\rho)^2]|^2_{\mathbb{L}^1}\\
&\quad\leqslant C(N)\,|\rho|^2_{\mathbb{L}^2}\,(|u|^4_{\mathbb{L}^4}+|\rho|^4_{\mathbb{L}^4})\\
&\quad\leqslant\tfrac{1}{2}C(N)\,|\rho|^2_{\mathbb{L}^2}\,(|u|^4_{\mathbb{L}^2}+|u|^2_{\mathbb{L}^2}\,|\nabla u|^2_{\mathbb{L}^2}+2\,|\rho|^4_{\mathbb{L}^4}).
\end{aligned}\tag{30}$$

Notice that if $L^t\in D(C)$ then for all $t\in[0,T]$

$$|u(t)|^2_{\mathbb{L}^2}<C_0+(2k_0+\mathscr{E}_0+\mathscr{E}_1)\,t+Ct^{\frac{4}{5}},$$

$$|\nabla u(t)|^2_{\mathbb{L}^2}<C_0+(2k_0+\mathscr{E}_0+\mathscr{E}_1)\,t+Ct^{\frac{4}{5}}.$$

In addition, we can apply the same analysis as in Section 2.1 to obtain

$$|\rho(t)|^2_{\mathbb{L}^2}\leqslant|\rho(0)|^2_{\mathbb{L}^2}\exp\left\{\left(-2\left[\frac{N}{2}\right]^2+2\right)t\right\}\leqslant 4C_0\exp\left\{\left(-2\left[\frac{N}{2}\right]^2+2\right)t\right\}.$$

For $|\rho|_{\mathbb{L}^4}^4$, we have

$$\frac{1}{4}\frac{d|\rho(t)|_{\mathbb{L}^4}^4}{dt} = \langle \Delta\rho, \rho^3\rangle_{\mathbb{L}^2} + |\rho|_{\mathbb{L}^4}^4 - \langle \rho[u^2 + u(u-\rho) + (u-\rho)^2], \rho^3\rangle_{\mathbb{L}^2}$$

$$\leqslant -3\langle \nabla\rho, \rho^2\,\nabla\rho\rangle_{\mathbb{L}^2} + |\rho|_{\mathbb{L}^4}^4 \leqslant |\rho|_{\mathbb{L}^4}^4.$$

Thus

$$|\rho(t)|_{\mathbb{L}^4}^4 \leqslant |\rho(0)|_{\mathbb{L}^4}^4 \exp(4t) \leqslant 16C_0^2 \exp(4t).$$

By assumption that $N > 3$, $|\rho(t)|_{\mathbb{L}^2}^2$ and $|\rho(t)|_{\mathbb{L}^2}^2\,|\rho(t)|_{\mathbb{L}^4}^4$ go to zero exponentially fast when $L^t \in D(C)$ and hence the estimate on the right hand side of (30) decays exponentially fast. Thus,

$$\sup_{\{\omega:\, \mathsf{S}_t^\omega L^0 \in D(C)\}} \int_0^\infty |D(\ell(t), \Phi_t(L^t, \Phi_0(L^0)), \Phi_t(L^t, \bar{h}(0)))|_{\mathbb{L}^2}^2\, dt < K(C) < \infty$$

for some constant $K(C)$. Thus Condition 3 holds.

We now move to Condition 4. Fix $L^0 \in \mathscr{P}$. Before continuing let us assume without loss of generality that $|\ell(0)|_{\mathbb{L}^2} \leqslant C_0$ and $t \leqslant T$ for some positive C_0 and T. Define

$$D_T(b_0) = \left\{ f \in C([0,\infty), \mathbb{L}_\ell^2) : \int_0^t |v(r)|_{\mathbb{L}^2}^6\, dr < (b_0 C_0)^6\, T \text{ for } 0 \leqslant t \leqslant T, \right.$$
$$\left. \text{where } v(s) = f(s) + \Phi_s(f, \Phi_0(L^0)) \right\}.$$

By Lemma 3.1, which says $|u|_{\mathbb{L}^2}^2$ grows polynomially on arbitrarily large sets, $\mathbb{P}\{\omega: \mathsf{S}_t^\omega L^0 \in D_T(b_0)\}$ can be made as close as we wish to 1 by increasing b_0. We will show that

$$\sup_{L^t \in D_T} \int_0^t |G(\ell(s), \Phi_s(L^s, \Phi_0(L^0)))|_{\mathbb{L}^2}^2\, ds < \infty.$$

Let $h(s) = \Phi_s(L^s, \Phi_0(L^0))$, then we have the following estimate on G:

$$|G(\ell(s), \Phi_s(L^s, h_0))|_{\mathbb{L}^2}$$
$$= \sup_{w \in \mathbb{L}^2,\, |w|_{\mathbb{L}^2} = 1} \langle \ell - (\ell + h)^3, P_\ell w\rangle$$
$$\leqslant |\ell|_{\mathbb{L}^2} + C \sup_{w \in \mathbb{L}^2,\, |w|_{\mathbb{L}^2} = 1} (|P_\ell\, \nabla w|_{\mathbb{L}^2}^2 + |P_\ell w|_{\mathbb{L}^2}^2)^{\frac{1}{2}} \langle |\ell + h|^2, |\ell + h|\rangle$$
$$\leqslant \tfrac{1}{3}\, |\ell|_{\mathbb{L}^2}^3 + 1 + C(N)(\big||h(s)|^{\frac{3}{2}}\big|_{\mathbb{L}^2}^2 + \big||\ell|^{\frac{3}{2}}\big|_{\mathbb{L}^2}^2).$$

By Sobolev inequality,

$$\left\| |\ell|^{\frac{3}{2}} \right|_{\mathbb{L}^2}^2 \leqslant |\ell|_{\mathbb{L}^\infty} \, |\ell|_{\mathbb{L}^2}^2 \leqslant \frac{1}{\sqrt{2}} \, (|\ell|_{\mathbb{L}^2}^2 + |\nabla \ell|_{\mathbb{L}^2}^2)^{\frac{1}{2}} \, |\ell|_{\mathbb{L}^2}^2 \leqslant \hat{C}(N) \, |\ell|_{\mathbb{L}^2}^3.$$

By Lemma 4.4 we know that if L^t is in D_T then $\sup_{s \in [0,t]} |h(t)|_{\mathbb{L}^2}$ is less than some C_1, where C_1 depends on $|h_0|_{\mathbb{L}^2}$ and the b_0, C_0 and T used to define D_T. Hence for any $\ell \in D_T$, we have

$$\begin{aligned} \int_0^t |G(\ell(s), \Phi_s(L^s, h_0))|_{\mathbb{L}^2}^2 \, ds &\leqslant C' \int_0^t [|\ell(s)|_{\mathbb{L}^2}^6 + \left\| h(s)|^{\frac{3}{2}} \right|_{\mathbb{L}^2}^4 + 1] \, ds \\ &\leqslant C'(b_0 C_0)^6 \, T + C'' C_1^6 t + C' t. \end{aligned}$$

Thus Condition 4 is satisfied.

Then we have the following theorem:

Theorem 2. For $N > 3$, the stochastic Ginzburg–Landau equation (22) has a unique invariant measure.

In [EH00], Eckman and Hairer proved uniqueness of the invariant measure for the stochastically forced Ginzburg–Landau equation when all but a few low modes are forced. J. Mattingly has informed us that he has also obtained the same result as in Theorem 2 using similar ideas.

3.2. Kuramoto–Sivashinsky Equation

We assume the initial condition and the random perturbation to be odd in the stochastic Kuramoto–Sivashinsky equation, which is equivalent to the no-slip boundary condition on $[0, \pi]$. Hence the solution is also odd. The same results for the general case is promising if we combine the technique in [CEES] and [G] with the strategy here. But this has not been achieved.

Thus we can discuss this problem in $\dot{\mathbb{H}}$, the space of all odd functions in $\mathbb{L}^2$. Then $\{\sin(kx), k \in N\}$ gives a basis for $\dot{\mathbb{H}}$. Let $\dot{\mathbb{H}}^\alpha = \dot{\mathbb{H}} \cap \mathbb{H}^\alpha$ denote the subspaces of odd functions of $\mathbb{H}^\alpha$. Let us consider the Schrödinger operator on $\mathbb{L}^2[-\pi, \pi]$:

$$\mathbb{K} = \Delta^2 w - q w, \tag{31}$$

where q is in $\dot{C}^\infty_{\text{per}} = \{\psi \in C^\infty, \psi(x) = \psi(x + 2\pi), \int_{-\pi}^{\pi} \psi(x) \, dx = 0\}$. $\mathbb{K}$ acts on $\mathbb{L}^2$ and its domain is $\mathbb{H}^4$. If q is an even function we observe that $\mathbb{K}$ maps $\dot{\mathbb{H}}^4$ into $\dot{\mathbb{H}}$, and we denote by $\mathbb{K}_0$ its restriction to $\dot{\mathbb{H}}$ with domain $D(\mathbb{K}_0) = \dot{\mathbb{H}}^4$. The proof of the following lemma can be found in [T97].

Lemma 3.2. For any $\alpha > 0$, there exists an even function q in $\dot{C}^{\infty}_{\text{per}}$ such that

$$(\mathbb{K}_0 w, w) \geqslant \tfrac{1}{2} |\Delta w|^2_{\mathbb{L}^2} + \alpha\, |w|^2_{\mathbb{L}^2}. \tag{32}$$

Suppose u is the solution of the stochastic Kuramoto–Sivashinsky equation (23). Let $u = w + \varphi$, where φ is an odd function such that $q = -\frac{1}{2} \nabla\varphi$ satisfies Lemma 3.2 with $\alpha = 2$. By integration, we can get φ from q. Then SKS equation (23) becomes:

$$dw(x, t) = -\Delta^2 w - \Delta w - \varphi\, \nabla w - w\, \nabla\varphi - w\, \nabla w + g(\varphi) + dW(t), \tag{33}$$

where $g(\varphi) = -\Delta^2\varphi - \Delta\varphi - \varphi\nabla\varphi$. We will discuss the SKS equation in the form of (33). We introduce the the following notations:

$$Aw = \Delta^2 w, \qquad R(w) = -\Delta w - \varphi\, \nabla w - w\, \nabla\varphi - w\, \nabla w + g(\varphi),$$

where $\{e_k(x)\} = \{\sin(kx)\}$, $k \in \mathbb{N}$ and $\lambda_k = k^4$.

By Lemma 3.2 and the way we choose φ, we have

$$-\tfrac{1}{2} \langle w\nabla\varphi, w\rangle_{\mathbb{L}^2} \leqslant \tfrac{1}{2} |\Delta w|^2_{\mathbb{L}^2} - 2\, |w|^2_{\mathbb{L}^2}.$$

And by interpolation

$$\begin{aligned} |\nabla w|^2_{\mathbb{L}^2} + \langle g(\varphi), w\rangle_{\mathbb{L}^2} &\leqslant |w|_{\mathbb{L}^2}\, |\Delta w|_{\mathbb{L}^2} + |g(\varphi)|_{\mathbb{L}^2}\, |w|_{\mathbb{L}^2} \\ &\leqslant \tfrac{1}{4} |\Delta w|^2_{\mathbb{L}^2} + 2\, |w|^2_{\mathbb{L}^2} + \tfrac{1}{4} |g(\varphi)|^2_{\mathbb{L}^2}. \end{aligned}$$

Therefore

$$\begin{aligned} -\langle \Delta^2 w, w\rangle_{\mathbb{L}^2} + \langle R(w), w\rangle_{\mathbb{L}^2} &= -|\Delta w|^2_{\mathbb{L}^2} + |\nabla w|^2_{\mathbb{L}^2} - \tfrac{1}{2} \langle w\, \nabla\varphi, w\rangle_{\mathbb{L}^2} + \langle g(\varphi), w\rangle_{\mathbb{L}^2} \\ &\leqslant -\tfrac{1}{4} |\Delta w|^2_{\mathbb{L}^2} + \tfrac{1}{4} |g(\varphi)|^2_{\mathbb{L}^2}, \end{aligned}$$

which means the SKS equation satisfies the Condition 1 with $\eta = \frac{1}{4}$ and $k_0 = \frac{1}{4} |g(\varphi)|^2_{\mathbb{L}^2}$. Moreover, we have:

Lemma 3.3. $\forall \delta > \frac{1}{2}$, $a \in (0, 1)$ and $C_0 > 0$, $\exists C(\delta, a, C_0) > 0$ such that if $|w_0|^2_{\mathbb{L}^2} < C_0$,

$$\mathbb{P}\left\{ |w(t)|^2_{\mathbb{L}^2} + \tfrac{1}{2} \int_0^t |\Delta w(s)|^2_{\mathbb{L}^2}\, ds \leqslant C_0 + C_1 t + C(t+1)^{\delta} \text{ for all } t \geqslant 0 \right\} \geqslant 1 - a,$$

where $C_1 = \frac{1}{2} |g(\varphi)|^2_{\mathbb{L}^2} + \mathscr{E}_0$.

Proof. The energy equation reads

$$|w(t)|_{\mathbb{L}^2}+\tfrac{1}{2}\int_0^t |\Delta w(s)|^2_{\mathbb{L}^2}\,ds \leqslant |w_0|^2_{\mathbb{L}^2}+\left(\tfrac{1}{2}\,|g(\varphi)|^2_{\mathbb{L}^2}+\mathscr{E}_0\right)t+2\int_0^t \langle w(s), dW(s)\rangle_{\mathbb{L}^2}.$$

Let $M_t^w=\int_0^t \langle w(s), dW(s)\rangle_{\mathbb{L}^2}$. Notice that

$$[M^w, M^w]_t \leqslant \sigma^2_{\max}\int_0^t |w(s)|^2_{\mathbb{L}^2}.$$

By an argument similar to the proof of Lemma 3.1, we have the conclusion. ∎

Now we move to the Condition 2. Suppose $\rho=w_1(t)-w_2(t)\in\mathbb{H}_h$, then

$$\begin{aligned}\langle R(u_1)-R(u_2), \rho\rangle_{\mathbb{L}^2} &= \langle -\Delta\rho-P_h[\varphi\,\nabla\rho+\rho\,\nabla\varphi+w_1\,\nabla w_1-w_2\,\nabla w_2], \rho\rangle_{\mathbb{L}^2}\\ &= |\nabla\rho|^2_{\mathbb{L}^2}-\tfrac{1}{2}\langle\nabla\varphi, \rho^2\rangle+\tfrac{1}{2}\langle[\rho(w_1+w_2)], \nabla\rho\rangle\\ &= |\nabla\rho|^2_{\mathbb{L}^2}-\tfrac{1}{2}\langle\nabla\varphi, \rho^2\rangle+\tfrac{1}{2}\langle[\rho(2w_1-\rho)], \nabla\rho\rangle\\ &= |\nabla\rho|^2_{\mathbb{L}^2}-\tfrac{1}{2}\langle\nabla\varphi, \rho^2\rangle+\langle w_1, \rho\,\nabla\rho\rangle.\end{aligned}$$

By Lemma 3.2,

$$-\tfrac{1}{2}\langle\nabla\varphi, \rho^2\rangle \leqslant \tfrac{1}{2}\,|\Delta\rho|^2_{\mathbb{L}^2}-2\,|\rho|^2_{\mathbb{L}^2}.$$

Using Sobolev inequality, we have

$$\langle w_1, \rho\,\nabla\rho\rangle \leqslant |\rho|_{L^\infty}\,|w_1\,\nabla\rho|_{L^1}\leqslant |w_1|_{\mathbb{L}^2}\,|\nabla\rho|^2_{\mathbb{L}^2}.$$

So

$$\begin{aligned}\langle R(u_1)-R(u_2), \rho\rangle_{\mathbb{L}^2} &\leqslant \tfrac{1}{2}\,|\Delta\rho|^2_{\mathbb{L}^2}+|\nabla\rho|^2_{\mathbb{L}^2}-2\,|\rho|^2_{\mathbb{L}^2}+|w_1|_{\mathbb{L}^2}\,|\nabla\rho|^2_{\mathbb{L}^2}\\ &\leqslant \tfrac{1}{2}\,|\Delta\rho|^2_{\mathbb{L}^2}+|\Delta\rho|_{\mathbb{L}^2}\,|\rho|_{\mathbb{L}^2}-2\,|\rho|^2_{\mathbb{L}^2}+|w_1|_{\mathbb{L}^2}\,|\Delta\rho|_{\mathbb{L}^2}\,|\rho|_{\mathbb{L}^2}\\ &\leqslant \tfrac{3}{4}\,|\Delta\rho|^2_{\mathbb{L}^2}+2\,|w_1|^2_{\mathbb{L}^2}\,|\rho|^2_{\mathbb{L}^2}.\end{aligned}$$

By Lemma 4.3, we know that $|w|_{\mathbb{L}^2}$ is in $\mathbb{L}^2(\mu)$ and

$$\int |w|^2_{\mathbb{L}^2}\,d\mu \leqslant \tfrac{1}{2}\,|g(\varphi)|^2_{\mathbb{L}^2}+\mathscr{E}_0.$$

Thus the SKS equation satisfies the Condition 2 with $\alpha=\frac{3}{4}$, $K(w)=2\,|w|^2_{\mathbb{L}^2}$ and $\beta=|g(\varphi)|^2_{\mathbb{L}^2}+2\mathscr{E}_0$. Equation (17) is equivalent to $N^4>4(|g(\varphi)|^2_{\mathbb{L}^2}+2\mathscr{E}_0)$. From now on, we assume $N^4>4(|g(\varphi)|^2_{\mathbb{L}^2}+2\mathscr{E}_0)$.

To check Condition 3, fix $L^0 \in \mathscr{P}$ and $\bar{h}(0)$ a high mode initial value. Let $L^s = \mathsf{S}_s^\omega L^0$ and $\ell(s) = L^t(s)$ for $0 \leqslant s \leqslant t$. Then with probability one, $h(s) = \Phi_s(L^s)$ where $u(s) = (\ell(s), h(s))$. Fix a constant C_0 such that $|w(0)|_{\mathbb{L}^2}^2 = |L^0(0)|_{\mathbb{L}^2}^2 \leqslant C_0$. For any positive C we define

$$D(C) = \Big\{ f \in C([0,\infty), \mathbb{L}_\ell^2) :$$

$$|w(t)|_{\mathbb{L}^2}^2 + \tfrac{1}{2}\int_0^t |\Delta w(s)|_{\mathbb{L}^2}^2\, ds \leqslant C_0 + \left(\tfrac{1}{2}\,|g(\varphi)|_{\mathbb{L}^2}^2 + \mathscr{E}_0\right) t + Ct^{\frac{4}{5}},$$

$$\text{where } v(s) = f(s) + \Phi_s(f, \Phi_0(L^0)) \Big\}.$$

By Lemma 3.3, we know that for any $a \in (0,1)$ there exists a C such that

$$\mathbb{P}\{\omega \colon \mathsf{S}_t^\omega L^0 \in D(C)\} > 1 - a > 0.$$

Let $\bar{h}(s) = \Phi_s(L^s, \bar{h}(0))$, $\rho(s) = h(s) - \bar{h}(s)$, then $w = \ell + h = \ell + \bar{h} + \rho$ and we have

$$\begin{aligned}
|D(\ell(s), h(s), \bar{h}(s))|_{\mathbb{L}^2}^2 &= \tfrac{1}{4} \sup_{v \in \mathbb{L}^2,\, |v| = 1} |\langle P_\ell(\nabla \rho(2w - \rho)), v\rangle|^2 \\
&= \tfrac{1}{4} \sup_{v \in \mathbb{L}^2,\, |v| = 1} |\langle \rho(2w - \rho), \nabla P_\ell v\rangle|^2 \\
&\leqslant C \sup_{v \in \mathbb{L}^2,\, |v| = 1} (|\Delta P_\ell v|_{\mathbb{L}^2}^2)(|\rho|_{\mathbb{L}^2}^4 + |\rho|_{\mathbb{L}^2}^2\, |w|_{\mathbb{L}^2}^2) \\
&\leqslant C(N)(|\rho|_{\mathbb{L}^2}^4 + |\rho|_{\mathbb{L}^2}^2\, |w|_{\mathbb{L}^2}^2).
\end{aligned} \tag{34}$$

Notice that if $L^t \in D(C)$ then for all $t \in [0, T]$

$$|w(t)|_{\mathbb{L}^2}^2 < C_0 + \left(\tfrac{1}{2}\,|g(\varphi)|_{\mathbb{L}^2}^2 + \mathscr{E}_0\right) t + Ct^{\frac{4}{5}},$$

$$\int_0^t |w(s)|_{\mathbb{L}^2}^2\, ds \leqslant \int_0^t |\Delta w(s)|_{\mathbb{L}^2}^2\, ds \leqslant 2C_0 + \left(|g(\varphi)|_{\mathbb{L}^2}^2 + 2\mathscr{E}_0\right) t + 2Ct^{\frac{4}{5}}.$$

In addition, applying the same analysis as in Section 2.1, we have

$$\begin{aligned}
|\rho(t)|_{\mathbb{L}^2}^2 &\leqslant |\rho(0)|_{\mathbb{L}^2}^2 \exp\left\{ -\tfrac{1}{2} N^4 t + 4\int_0^t |w(s)|_{\mathbb{L}^2}^2\, ds \right\} \\
&\leqslant 4C_0 \exp\{-\tfrac{1}{2} N^4 t + 8C_0 + 4(|g(\varphi)|_{\mathbb{L}^2}^2 + 2\mathscr{E}_0)\, t + 8Ct^{\frac{4}{5}}\}.
\end{aligned}$$

Assume that $N^4 > 8(|g(\varphi)|^2_{\mathbb{L}^2} + 2\mathscr{E}_0)$, we see then the estimate on the right hand side of (34) decays exponentially fast when $L^t \in D(C)$. Thus,

$$\sup_{\omega:\, S_t^{\omega} L^0 \in D(C)} \int_0^{\infty} |D(\ell(t), \Phi_t(L^t, \Phi_0(L^0)), \Phi_t(L^t, \bar{h}(0)))|^2_{\mathbb{L}^2}\, dt < \text{const.}\ K(C) < \infty,$$

which implies that the SKS equation satisfies Condition 3 when $N^4 > 8(|g(\varphi)|^2_{\mathbb{L}^2} + 2\mathscr{E}_0)$.

To Condition 4, define D_T to be

$$D_T(b_0) = \Big\{ f \in C([0, \infty), \mathbb{L}^2_\ell) : \int_0^t |v(r)|^4_{\mathbb{L}^2}\, dr < (b_0 C_0)^4\, T \text{ for } 0 \leqslant t \leqslant T, \\ \text{where } v(s) = f(s) + \Phi_s(f, \Phi_0(L^0)) \Big\}$$

By Lemma 3.3, which says that $|w|^2_{\mathbb{L}^2}$ grows polynomially on arbitrarily large sets, $\mathbb{P}\{\omega\colon \mathsf{S}_t^{\omega} L^0 \in D_T(b_0)\}$ can be made as close as we wish to 1 by increasing b_0. We will show that

$$\sup_{L^t \in D_T} \int_0^t |G(\ell(s), \Phi_s(L^s, \Phi_0(L^0)))|^2_{\mathbb{L}^2}\, ds < \infty.$$

Let $\ell(s) = \Phi_s(L^s, h_0))$ where $h_0 = \Phi_0(L^0)$, we have the following estimate on G:

$$\begin{aligned} |G(\ell(s), \Phi_s(L^s, h_0))|_{\mathbb{L}^2} &= \sup_{w \in \mathbb{L}^2,\, |w|_{\mathbb{L}^2} = 1} |\langle \Delta(\ell + \varphi) + (h + \ell + \varphi)\, \nabla(h + \ell + \varphi), P_l w\rangle| \\ &\leqslant \sup_{w \in \mathbb{L}^2,\, |w|_{\mathbb{L}^2} = 1} |\langle \ell + \varphi, \Delta P_\ell w\rangle| + \tfrac{1}{2}\, |\langle (\ell + h + \varphi)^2, \nabla P_\ell w\rangle| \\ &\leqslant C(\varphi) \sup_{w \in \mathbb{L}^2,\, |w|_{\mathbb{L}^2} = 1} |P_\ell \nabla \cdot \Delta w|_{\mathbb{L}^2}\, (|h|^2_{\mathbb{L}^2} + |l|^2_{\mathbb{L}^2} + 1) \\ &\leqslant C(N, \varphi)(|h|^2_{\mathbb{L}^2} + |l|^2_{\mathbb{L}^2} + 1). \end{aligned}$$

By Lemma 4.5 and the fact that φ is a constant with respect to time, we know that if L^t is in D_T then $\sup_{s \in [0, t]} |h(t)|_{\mathbb{L}^2}$ is less than some C_1, where C_1 depends on $|h_0|_{\mathbb{L}^2}$ and the b_0, C_0 and T used to define D_T. Hence for any $\ell \in D_T$, we have

$$\begin{aligned} \int_0^t |G(\ell(s), \Phi_s(L^s, h_0))|^2_{\mathbb{L}^2}\, ds &\leqslant C' \int_0^t [|\ell(s)|^4_{\mathbb{L}^2} + |h(s)|^4_{\mathbb{L}^2} + 1]\, ds \\ &\leqslant C'(b_0 C_0)^4\, T + C'' C_1^4 t + C' t. \end{aligned}$$

So the SKS equation satisfies Condition 4.

Then we can conclude the theorem for the SKS equation:

Theorem 3. For $N^4 > 8(|g(\varphi)|^2_{\mathbb{L}^2} + 2\mathscr{E}_0)$, the stochastic Kuramoto–Sivashinsky equation (23) has a unique invariant measure.

3.3. Cahn–Hilliard Equation

We assume that the random perturbation and the initial condition have zero means in the stochastic Cahn–Hilliard equation. As a consequence, the solution also has zero mean. Define $\mathbb{H}^\alpha = \{v \in \mathbb{H}^\alpha$, and $\int_{-\pi}^{\pi} v(x)\,dx = 0\}$, the subspace of $\mathbb{H}^\alpha$ with zero means. We will work on space $\mathbb{H}^0$ with the following notations:

$$Au = \Delta^2 u, \qquad R(u) = \Delta V'(u).$$

Then $\{e_k\} = \{\frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \ldots, \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}, \ldots\}$, $\lambda_k = [\frac{n+1}{2}]^4$, $k \in \mathbb{N}$. We suppose that $V(u)$ is twice continuously differentiable and satisfies the following condition:

$$b = \sup |V''(u)| < 1.$$

Then we have

$$\begin{aligned} -\langle Au, u\rangle_{\mathbb{L}^2} + \langle R(u), u\rangle_{\mathbb{L}^2} &= -|\Delta u|^2_{\mathbb{L}^2} - \langle V''(u)\,\nabla u, \nabla u\rangle_{\mathbb{L}^2} \\ &\leqslant \sup_{k \in N} (-k^4 + bk^2)\,|u|^2_{\mathbb{L}^2} \\ &\leqslant (-1+b)\,|u|^2_{\mathbb{L}^2}. \end{aligned}$$

So the SCH equation satisfies the Condition 1 with $\eta = 1 - b$ and $k_0 = 0$.

For Condition 2, by the same argument

$$\begin{aligned} \langle R(u_1) - R(u_2), \rho\rangle_{\mathbb{L}^2} &= \langle \Delta(V'(u_1) - V'(u_2)), \rho\rangle_{\mathbb{L}^2} \\ &\leqslant \sup |V''(\cdot)|\,|\nabla\rho|^2_{\mathbb{L}^2} \\ &\leqslant b\,|\nabla\rho|^2_{\mathbb{L}^2}. \end{aligned}$$

By Poincaré inequality, we know the SCH equation satisfies Condition 2 with $\alpha = b$ and $K(u) = 0$. Equation (17) is equivalent to $N \geqslant 1$.

Lemma 3.4. $\forall \delta > \frac{1}{2}$, $a \in (0, 1)$ and $C_0 > 0$, $\exists C(\delta, a, C_0) > 0$ such that if $|u_0|^2_{\mathbb{L}^2} < C_0$,

$$\mathbb{P}\left\{ |u|^2_{\mathbb{L}^2} + 2(1-b)\int_0^t |u(s)|^2_{\mathbb{L}^2}\, ds \leqslant C_0 + \mathscr{E}_0 t + C(t+1)^{\delta} \text{ for all } t \geqslant 0 \right\} \geqslant 1-a.$$

Proof. Applying Ito's formula to $u(t) \mapsto |u(t)|^2_{\mathbb{L}^2}$, we have

$$\begin{aligned} d|u|^2_{\mathbb{L}^2} &= 2[-|\Delta u|^2_{\mathbb{L}^2} + \langle \Delta V'(u), u\rangle_{\mathbb{L}^2}]\, dt + 2\langle u, dW\rangle_{\mathbb{L}^2} + \mathscr{E}_0\, dt \\ &\leqslant 2(b-1)\, |u|^2_{\mathbb{L}^2}\, dt + 2\langle u, dW\rangle_{\mathbb{L}^2} + \mathscr{E}_0\, dt. \end{aligned}$$

By Corollary 4.2 and an argument similar to the proof of Lemma 3.1, we can have the conclusion. ∎

Now we go to the Condition 3 for SCH equation. Fix $L^0 \in \mathscr{P}$ and $\bar{h}(0)$ a high mode initial value. Let $L^s = \mathsf{S}^{\omega}_s L^0$ and $\ell(s) = L^t(s)$ for $s \leqslant t$. Then with probability one, $h(s) = \Phi_s(L^s)$ where $u(s) = (\ell(s), h(s))$. It would be enough to show that

$$\sup_{\omega} \int_0^{\infty} |D(\ell(t), \Phi_t(L^t, \Phi_0(L^0)), \Phi_t(L^t, \bar{h}(0)))|^2_{\mathbb{L}^2}\, dt < \infty.$$

Putting $\bar{h}(s) = \Phi_s(L^s, \bar{h}(0))$, $\rho(s) = h(s) - \bar{h}(s)$, then $u = \ell + h = \ell + \bar{h} + \rho$, and we have

$$\begin{aligned} |D(\ell(s), h(s), \bar{h}(s))|^2_{\mathbb{L}^2} &= \sup_{w \in \mathbb{L}^2, |w|=1} |\langle P_\ell \Delta(V'(u) - V'(u-\rho)), w\rangle|^2 \\ &= \sup_{w \in \mathbb{L}^2, |w|=1} |\langle (V'(u) - V'(u-\rho), \Delta P_\ell w\rangle|^2 \\ &\leqslant \sup_{w \in \mathbb{L}^2, |w|=1} |\Delta P_\ell w|^2_{\mathbb{L}^2}\, |\rho|^2_{\mathbb{L}^2} \\ &\leqslant C(N)\, |\rho|^2_{\mathbb{L}^2}. \end{aligned} \tag{35}$$

While ρ satisfies the following equation:

$$d\rho = [-\Delta^2 \rho + P_h \Delta(V'(u_1) - V'(u_2))]\, dt.$$

So by the same argument in Condition 2,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} |\rho|^2_{\mathbb{L}^2} &= -|\Delta\rho|^2_{\mathbb{L}^2} - \langle \nabla(V'(u_1) - V'(u_2)), \nabla\rho\rangle \\ &\leqslant -|\Delta\rho|^2_{\mathbb{L}^2} + b\, |\nabla\rho|^2_{\mathbb{L}^2} \leqslant (-1+b)\, |\rho|^2_{\mathbb{L}^2}. \end{aligned}$$

Hence

$$|\rho(t)|_{\mathbb{L}^2}^2 \leqslant |\rho(0)|_{\mathbb{L}^2}^2 \exp\{2(b-1)\,t\} \leqslant 4C_0 \exp\{2(b-1)\,t\}.$$

Thus the right hand side of (35) decays exponentially fast. Thus,

$$\sup_{\omega} \int_0^{\infty} |D(\ell(t), \Phi_t(L^t, \Phi_0(L^0)), \Phi_t(L^t, \bar{h}(0)))|_{\mathbb{L}^2}^2 \, dt < \infty,$$

which implies Condition 3 for the SCH equation.

For Condition 4, first define D_T to be

$$D_T(b_0) = \left\{ f \in C([0, \infty), \mathbb{L}_\ell^2) : \int_0^t |v(r)|_{\mathbb{L}^2}^2 \, dr < (b_0 C_0)^2 \, T \text{ for } 0 \leqslant t \leqslant T, \right.$$
$$\left. \text{where } v(s) = f(s) + \Phi_s(f, \Phi_0(L^0)) \right\}.$$

By Lemma 3.4, which says $|u|_{\mathbb{L}^2}^2$ grows polynomially on arbitrarily big sets, $\mathbb{P}\{\omega\colon \mathsf{S}_t^{\omega} L^0 \in D_T(b_0)\}$ can be made as close as enough to 1 by increasing b_0. We will prove that

$$\sup_{L^t \in D_T} \int_0^t |G(\ell(s), \Phi_s(L^s, \Phi_0(L^0)))|_{\mathbb{L}^2}^2 \, ds < \infty.$$

Let $h(s) = \Phi_s(L^s, h_0))$ and $h_0 = \Phi_0(L^0))$, we have the following estimate on G:

$$\begin{aligned} |G(\ell(s), \Phi_s(L^s, h_0))|_{\mathbb{L}^2}^2 &= \sup_{w \in \mathbb{L}^2, |w|_{\mathbb{L}^2} = 1} |\langle P_\ell \Delta V'(h+\ell), w \rangle|^2 \\ &= \sup_{w \in \mathbb{L}^2, |w|_{\mathbb{L}^2} = 1} |\langle V'(h+\ell), \Delta P_\ell w \rangle|^2 \\ &\leqslant \sup_{w \in \mathbb{L}^2, |w|_{\mathbb{L}^2} = 1} |P_\ell \nabla \cdot \Delta w|_{\mathbb{L}^2}^2 \, (|h+\ell|_{\mathbb{L}^1} + V'(0)) \\ &\leqslant C(N)(|h|_{\mathbb{L}^2} + |\ell|_{\mathbb{L}^2} + c). \end{aligned}$$

By Lemma 4.6 we know that if L^t is in D_T then $\sup_{s \in [0, t]} |h(t)|_{\mathbb{L}^2}$ is less than some C_1, where C_1 depends on $|h_0|_{\mathbb{L}^2}$ and the b_0, C_0 and T used to define D_T. Hence for any $L^t \in D_T$, we have

$$\begin{aligned} \int_0^t |G(\ell(s), \Phi_s(L^s, h_0))|_{\mathbb{L}^2}^2 \, ds &\leqslant C' \int_0^t [|\ell(s)|_{\mathbb{L}^2}^2 + |h(s)|_{\mathbb{L}^2}^2 + c] \, ds \\ &\leqslant C'(b_0 C_0)^2 \, T + C'' C_1^2 t + C' c t. \end{aligned}$$

Hence the Condition 4 is satisfied by the SCH equation.

Now we have the theorem for the stochastic Cahn–Hilliard equation:

Theorem 4. For $N \geqslant 1$, the stochastic Cahn–Hilliard equation has a unique invariant measure.

4. ESTIMATES

4.1. Energy Estimates

In this section, we will give some energy estimates for general stochastic dissipative PDEs under Condition 1. As before, define the constants $\mathscr{E}_0 = \sum |\sigma_k|^2$, $\sigma_{\max}^2 = \sup |\sigma_k|^2$ and $(\Delta\sigma_{\max})^2 = \sup |k^2\sigma_k|^2$.

Lemma 4.1. For any $p \geqslant 1$, we have

$$\mathbb{E}\,|u(t)|_{\mathbb{H}}^{2p} + 2\eta p \int_0^t \mathbb{E}\,|u(s)|_{\mathbb{H}}^{2p}\, ds \leqslant \mathbb{E}\,|u(0)|_{\mathbb{H}}^{2p} + C_0 \int_0^t \mathbb{E}\,|u(s)|_{\mathbb{H}}^{2(p-1)}\, ds, \tag{36}$$

where $C_0 = 2p(p-1)\,\sigma_{\max}^2 + p(2k_0 + \mathscr{E}_0)$.

Proof. Applying Itô's formula to the map $u(t) \mapsto |u(t)|_{\mathbb{H}}^{2p}$ and using Condition 1, we have

$$\begin{aligned} d|u(t)|_{\mathbb{H}}^{2p} &= 2p\,|u(t)|_{\mathbb{H}}^{2(p-1)}\,[-\langle Au(t), u(t)\rangle_{\mathbb{H}}\, dt + \langle R(u(t)), u(t)\rangle_{\mathbb{H}}\, dt + \langle u(t), dW\rangle_{\mathbb{H}}] \\ &\quad + 2p(p-1)\,|u(t)|_{\mathbb{H}}^{2(p-2)} \left(\sum_k |u_k(t)|^2\,|\sigma_k|^2 \right) dt + p\,|u(t)|_{\mathbb{H}}^{2(p-1)}\,\mathscr{E}_0\, dt \\ &\leqslant 2p\,|u(t)|_{\mathbb{H}}^{2(p-1)}\,[-\eta\,|u|_{\mathbb{H}}^2\, dt + k_0\, dt + \langle u(t), dW\rangle_{\mathbb{H}}] \\ &\quad + 2p(p-1)\,\sigma_{\max}^2\,|u(t)|_{\mathbb{H}}^{2(p-1)}\, dt + p\,|u(t)|_{\mathbb{H}}^{2(p-1)}\,\mathscr{E}_0\, dt. \end{aligned} \tag{37}$$

For a fixed $H > 0$, define the stopping time T to be

$$T = \inf\{t \geqslant 0 : |u(t)|_{\mathbb{H}}^2 \geqslant H^2\}.$$

Denoting by M_t the local martingale term in (37), define

$$M_t^T = \int_0^t 2p\,|u(s\wedge T)|_{\mathbb{H}}^{2(p-1)}\,\langle u(s\wedge T), dW(s)\rangle_{\mathbb{H}}.$$

Let $[M^T, M^T]_t$ to be the quadratic variation of M_t^T, then

$$[M^T, M^T]_t \leqslant 4p^2\sigma_{\max}^2 \int_0^t |u(s\wedge T)|_{\mathbb{H}}^{4p-2}\, ds \leqslant 4p^2\sigma_{\max}^2 H^{4p-2} t < \infty.$$

So $\mathbb{E}[M^T, M^T]_t < \infty$. Hence M_t^T is a martingale and $\mathbb{E}M_t^T = 0$. By the Optional Sampling theorem we have $\mathbb{E}M_{t\wedge T}^T = 0$. Since $M_{t\wedge T} = M_{t\wedge T}^T$, we have

$$\mathbb{E}\,|u(t\wedge T)|_{\mathbb{H}}^{2p} + 2\eta p\mathbb{E}\int_0^{t\wedge T} |u(s)|_{\mathbb{H}}^{2p}\,ds$$
$$\leqslant \mathbb{E}\,|u(0)|_{\mathbb{H}}^{2p} + [2p(p-1)\,\sigma_{\max}^2 + p(2k_0 + \mathscr{E}_0)]\,\mathbb{E}\int_0^{t\wedge T} |u(s)|_{\mathbb{H}}^{2(p-1)}\,ds.$$

Since $u(t)$ is continuous in time, $T \to \infty$ as $H \to \infty$ and hence $T \wedge t \to t$. Thus we obtain

$$\mathbb{E}\,|u(t)|_{\mathbb{H}}^{2p} + 2\eta p\mathbb{E}\int_0^{t} |u(s)|_{\mathbb{H}}^{2p}\,ds$$
$$\leqslant \mathbb{E}\,|u(0)|_{\mathbb{H}}^{2p} + [2p(p-1)\,\sigma_{\max}^2 + p(2k_0 + \mathscr{E}_0)]\,\mathbb{E}\int_0^{t} |u(s)|_{\mathbb{H}}^{2(p-1)}\,ds. \quad \blacksquare$$

By Gronwall's inequality, we have the following estimates uniformly in time.

Corollary 4.2.

$$\mathbb{E}\,|u(t)|_{\mathbb{H}}^2 \leqslant e^{-2\eta t}\mathbb{E}\,|u(0)|_{\mathbb{H}}^2 + \left(\frac{2k_0 + \mathscr{E}_0}{2\eta}\right)(1 - e^{-2\eta t}). \tag{38}$$

And for any p > 1

$$\mathbb{E}\,|u(t)|_{\mathbb{H}}^{2p} \leqslant e^{-2\eta p t}\mathbb{E}\,|u(0)|_{\mathbb{H}}^{2p} + C_0\int_0^t e^{-2\eta p(t-s)}\mathbb{E}\,|u(s)|_{\mathbb{H}}^{2(p-1)}\,ds. \tag{39}$$

Now we establish a number of properties, derived from Corollary 4.2, that invariant measures for general SPDEs of the form (6) must have.

Lemma 4.3. Let μ be an invariant measure on $\mathbb{H}$ for Eq. (6). Then for any $p \geqslant 1$ there exists a constant $C_p < \infty$ such that

$$\int_{\mathbb{H}} |u|_{\mathbb{H}}^{2p}\,d\mu(u) < C_p. \tag{40}$$

Proof. Suppose $p=1$. Then $\forall\epsilon>0$, $\exists b_\epsilon$ such that $\mu\{u\in\mathbb{H}:|u|^2_{\mathbb{H}}\leqslant b_\epsilon\}>1-\epsilon$. Let $B_\epsilon=\{u\in\mathbb{H}:|u|^2_{\mathbb{H}}\leqslant b_\epsilon\}$. Then $\forall H>0$ and $t>0$, we have

$$\int_{\mathbb{H}}(|u|^2_{\mathbb{H}}\wedge H)\,d\mu(u)=\int_{\mathbb{H}}\mathbb{E}(|\varphi^{\omega}_{0,t}u|^2_{\mathbb{H}}\wedge H)\,d\mu(u)\leqslant H\epsilon+\int_{B_\epsilon}\mathbb{E}(|\varphi^{\omega}_{0,t}u|^2_{\mathbb{H}})\,d\mu(u).$$

Applying the first bound (38) in Corollary 4.2 gives

$$\int_{\mathbb{H}}(|u|^2_{\mathbb{H}}\wedge H)\,d\mu(u)\leqslant H\epsilon+\frac{(2k_0+\mathscr{E}_0)}{2\eta}+e^{-2\eta t}\left(b_\epsilon-\frac{(2k_0+\mathscr{E}_0)}{2\eta}\right).$$

Let $t\to\infty$ and notice that ϵ was arbitrary, we obtain

$$\int_{\mathbb{H}}(|u|^2_{\mathbb{H}}\wedge H)\,d\mu(u)\leqslant\frac{2k_0+\mathscr{E}_0}{2\eta}.$$

Let $H\to\infty$, we obtain (40) for $p=1$. The argument for higher moments of the energy is the same. ▮

Now we give the proof of Lemma 2.1 claimed in Section 2.

Proof of Lemma 2.1. The basic energy estimate, derived from (37), reads:

$$|u(t)|^2_{\mathbb{H}}\leqslant|u(t_0)|^2_{\mathbb{H}}+(2k_0+\mathscr{E}_0)(t-t_0)-2\eta\int_{t_0}^{t}|u(s)|^2_{\mathbb{H}}\,ds+2\int_{t_0}^{t}\langle u(s),dW(s)\rangle_{\mathbb{L}^2}.$$

For any $k\geqslant 1$, the above estimate implies

$$\sup_{s\in[-k,-k+1]}|u(s)|^2_{\mathbb{H}}\leqslant|u(-k)|^2_{\mathbb{H}}+2k_0+\mathscr{E}_0+\sup_{s\in[-k,-k+1]}F_k(s),$$

where $F_k(s)=-2\eta\int_{-k}^{s}|u(r)|^2_{\mathbb{H}}\,dr+2M_k(s)$ and $M_k(s)=\int_{-k}^{s}\langle u(r),dW(r)\rangle_{\mathbb{H}}$.
Now define

$$A_k=\{u(s):\sup_{s\in[-k,-k+1]}|u(s)|^2_{\mathbb{H}}\leqslant 2k_0+\mathscr{E}_0+K_0\,|k-1|^\delta\}.$$

By Borel–Cantelli lemma, we need only to show that $\sum_{k>0}\mu_p(A_k^c)<\infty$.
Notice that

$$\mu_p(A_k^c)\leqslant\mu_p\left\{u(s):|u(-k)|^2_{\mathbb{H}}\geqslant\frac{K_0}{2}\,|k-1|^\delta\right\}$$
$$+\mu_p\left\{u(s):\sup_{s\in[-k,-k+1]}F_k(s)\geqslant\frac{K_0}{2}\,|k-1|^\delta\right\}.$$

Lemma 4.3 implies that the second moment of the energy under the invariant measure is uniformly bounded by some constant C_2. Hence Chebyshev's inequality produces

$$\mu_p\left\{u(s)\colon |u(-k)|_{\mathbb{H}}^2 \geqslant \frac{K_0}{2}|k-1|^\delta\right\} \leqslant \frac{4}{K_0^2|k-1|^{2\delta}}\int_{\mathbb{H}} |u(-k)|_{\mathbb{H}}^4 \leqslant \frac{4C_2}{K_0^2|k-1|^{2\delta}},$$

which is summable as long as $\delta > \frac{1}{2}$.

For the second term, first notice that with probability one,

$$[M_k, M_k](s) = \int_{-k}^{s} \sum_l |\sigma_l|^2 |u_l(r)|^2 \, dr \leqslant \sigma_{\max}^2 \int_{-k}^{s} |u(r)|_{\mathbb{H}}^2 \, dr.$$

And hence

$$F_k(s) \leqslant 2M_k(s) - \frac{2\eta}{\sigma_{\max}^2}[M_k, M_k](s)$$

almost surely. And the exponential martingale inequality says that for positive α and β,

$$\mathbb{P}\left\{\sup_{s\in[-k,0]} M_k(s) - \frac{\alpha}{2}[M_k, M_k](s) > \beta\right\} \leqslant e^{-\alpha\beta}.$$

Taking $\alpha = \frac{2\eta}{\sigma_{\max}^2}$ and $\beta = \frac{K_0}{4}|k-1|^\delta$ we find

$$\mu_p\left\{u(s)\colon \sup_{s\in[-k,-k+1]} F_k(s) \geqslant \frac{K_0}{2}|k-1|^\delta\right\} \leqslant \exp\left(-\frac{\eta K_0}{2\sigma_{\max}^2}|k-1|^\delta\right).$$

Since this is summable for any $\delta > 0$, the proof is complete. ∎

4.2. Control of High Modes

4.2.1. Ginzburg–Landau Equation

Lemma 4.4. If $h(t)$ is the solution to (11) of the SGL equation with some low mode forcing $\ell \in C([0,t], \mathbb{L}_\ell^2)$, then $\sup_{s\in[0,t]} |h(s)|_{\mathbb{L}^2}$ is bounded by a constant depending on $|h(0)|_{\mathbb{L}^2}$ and $\int_0^t |\ell|_{\mathbb{L}^2}^4 \, ds$.

Proof. Taking the inner product of (11) with h produces

$$\frac{1}{2}\frac{d}{dt}|h(t)|_{\mathbb{L}^2}^2 = -|\nabla h|_{\mathbb{L}^2}^2 + |h|_{\mathbb{L}^2}^2 - \langle P_h(l+h)^3, h\rangle$$

$$= -|\nabla h|_{\mathbb{L}^2}^2 + |h|_{\mathbb{L}^2}^2 - \langle l^3, h\rangle - 3\langle l^2 h, h\rangle - 3\langle l h^2, h\rangle - \langle h^2, h^2\rangle.$$

Since

$$h^4+3lh^3+\tfrac{9}{4}l^2h^2 \geqslant 0, \qquad \tfrac{3}{4}l^2h^2+l^3h+\tfrac{1}{3}l^4 \geqslant 0,$$

we have

$$\frac{1}{2}\frac{d}{dt}|h(t)|^2_{\mathbb{L}^2} \leqslant |h|^2_{\mathbb{L}^2}+\frac{1}{3}|l^2|^2_{\mathbb{L}^2} \leqslant |h|^2_{\mathbb{L}^2}+\frac{1}{3}|l|^2_{\mathbb{L}^\infty}|l|^2_{\mathbb{L}^2}$$

$$\leqslant |h|^2_{\mathbb{L}^2}+\frac{1}{6}|l|^2_{\mathbb{L}^2}(|l|^2_{\mathbb{L}^2}+|\nabla l|^2_{\mathbb{L}^2}).$$

Since $\ell \in \mathbb{L}^2_\ell$ we have $|\nabla \ell|_{\mathbb{L}^2} \leqslant C(N)\,|\ell|_{\mathbb{L}^2}$ where $N=\sup\{|k|: \exists e_k \text{ with } e_k \in \mathbb{L}^2_\ell\}$, and hence after applying Gronwall's Lemma we have

$$|h(t)|^2_{\mathbb{L}^2} \leqslant |h(0)|^2_{\mathbb{L}^2}\exp(2t)+C_1\left(\int_0^t |\ell|^4_{\mathbb{L}^2}\,ds\right)\exp(2t). \quad \blacksquare$$

4.2.2. Kuramoto–Sivashinsky Equation

Lemma 4.5. If $h(t)$ is the solution to (11) in the SKS equation with some low mode forcing $\ell \in C([0,t],\mathbb{L}^2_\ell)$, then $\sup_{s\in[0,t]}|h(s)|_{\mathbb{L}^2}$ is bounded by a constant depending on $|h(0)|_{\mathbb{L}^2}$ and $\int_0^t |\ell|^4_{\mathbb{L}^2}\,ds$.

Proof. Taking the inner product of (11) with h produces

$$\frac{1}{2}\frac{d}{dt}|h(t)|^2_{\mathbb{L}^2} = -|\Delta h|^2_{\mathbb{L}^2}+|\nabla h|^2_{\mathbb{L}^2}-\langle P_h\nabla(\ell+h)^2, h\rangle$$

$$= -|\Delta h|^2_{\mathbb{L}^2}+|\nabla h|^2_{\mathbb{L}^2}-2\langle \ell\,\nabla\ell, h\rangle+2\langle \ell h, \nabla h\rangle.$$

Since

$$-|\Delta h|^2_{\mathbb{L}^2}+|\nabla h|^2_{\mathbb{L}^2} \leqslant 0, \qquad -2\langle \ell\,\nabla \ell, h\rangle \leqslant 2\,|\Delta\ell|_{\mathbb{L}^2}\,|\ell|_{\mathbb{L}^2}\,|h|_{\mathbb{L}^2} \leqslant |\Delta\ell|^2_{\mathbb{L}^2}\,|\ell|^2_{\mathbb{L}^2}+|h|^2_{\mathbb{L}^2}$$

and

$$2\langle \ell h, \nabla h\rangle = -\langle \nabla\ell, h^2\rangle \leqslant |\Delta\ell|_{\mathbb{L}^2}\,|h|^2_{\mathbb{L}^2},$$

we have

$$\frac{1}{2}\frac{d}{dt}|h(t)|^2_{\mathbb{L}^2} \leqslant |h|^2_{\mathbb{L}^2}+|\Delta\ell|_{\mathbb{L}^2}\,|h|^2_{\mathbb{L}^2}+|\Delta\ell|^2_{\mathbb{L}^2}\,|\ell|^2_{\mathbb{L}^2}.$$

And we also have $|\Delta l|_{\mathbb{L}^2} \leqslant C(N)\,|l|_{\mathbb{L}^2}$. Hence after applying Gronwall's Lemma we have

$$|h(t)|^2_{\mathbb{L}^2} \leqslant |h(0)|^2_{\mathbb{L}^2} \exp\left(2C\int_0^t |\ell|_{\mathbb{L}^2}\,ds+2t\right)$$
$$+C_1\left(\int_0^t |\ell|^4_{\mathbb{L}^2}\,ds\right)\exp\left(2C\int_0^t |\ell|_{\mathbb{L}^2}\,ds+2t\right).$$

By Hölder inequality $(\int_0^t |\ell|_{\mathbb{L}^2}\,ds)^4 \leqslant t^3 \int_0^t |\ell|^4_{\mathbb{L}^2}\,ds$, the proof is complete. ∎

4.2.3. Cahn–Hilliard Equation

Lemma 4.6. If $h(t)$ is the solution to (11) in the SCH equation with some low mode forcing $\ell \in C([0,t],\mathbb{L}^2_\ell)$, then $\sup_{s\in[0,t]} |h(s)|_{\mathbb{L}^2}$ is bounded by a constant depending on $|h(0)|_{\mathbb{L}^2}$ and $\int_0^t |\ell|^2_{\mathbb{L}^2}\,ds$.

Proof. Taking the inner product of (11) with h and making use of the assumption on V produce

$$\frac{1}{2}\frac{d}{dt}|h(t)|^2_{\mathbb{L}^2} = -|\Delta h|^2_{\mathbb{L}^2} - \langle \nabla V'(l+h), \nabla h\rangle = -|\Delta h|^2_{\mathbb{L}^2} - \langle V''(u)\,\nabla(l+h), \nabla h\rangle$$
$$\leqslant -|\Delta h|^2_{\mathbb{L}^2} + |\nabla h|^2_{\mathbb{L}^2} + |\nabla l|_{\mathbb{L}^2}\,|\nabla h|_{\mathbb{L}^2} \leqslant -|\Delta h|^2_{\mathbb{L}^2} + \frac{3}{2}|\nabla h|^2_{\mathbb{L}^2} + \frac{1}{2}|\nabla l|^2_{\mathbb{L}^2}.$$

Since $-|\Delta h_k|_{\mathbb{L}^2} + \frac{3}{2}|\nabla h_k|_{\mathbb{L}^2} = (-k^4+\frac{3}{2}k^2)\,|h_k|^2_{\mathbb{L}^2} \leqslant C_1\,|h_k|^2_{\mathbb{L}^2}$, for some constant C_1, we have $-|\Delta h|^2_{\mathbb{L}^2} + \frac{3}{2}|\nabla h|^2_{\mathbb{L}^2} \leqslant C_1\,|h|^2_{\mathbb{L}^2}$. And we also have $|\nabla l|^2_{\mathbb{L}^2} \leqslant C(N)\,|\ell|^2_{\mathbb{L}^2}$, and hence after applying Gronwall's Lemma we have

$$|h(t)|^2_{\mathbb{L}^2} \leqslant |h(0)|^2_{\mathbb{L}^2}\exp(2C_1 t) + C\left(\int_0^t |\ell|^2_{\mathbb{L}^2}\,ds\right)\exp(2C_1 t). \quad ∎$$

ACKNOWLEDGMENTS

We are very grateful to J. Mattingly and Ya. Sinai for many stimulating discussions concerning the topics here. Weinan E's work is partially supported by NSF via a PFF award.

REFERENCES

[BKL] J. Bricmont, A. Kupiainen, and R. Lefevere, Ergodicity of the 2D Navier–Stokes equations with random forcing, *Commun. Math. Phys.*, to appear.

[CEES] P. Collet, J.-P. Eckmann, H. Epstein, and J. Stubbe, A global attracting set for the Kuramoto–Sivashinsky equation, *Commun. Math. Phys.* **152**:203–214 (1993).

[DPZ92] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and Its Applications (Cambridge University Press, 1992).

[DPZ96] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, London Mathematics Society Lecture Note Series 229 (Cambridge University Press, 1996).

[EH00] J.-P. Eckmann and M. Hairer, *Uniqueness of the Invariant Measures for a Stochastic PDE Driven by Degenerate Noise*, preprint (2000).

[EMatt] W. E and J. C. Mattingly, Ergodicity for the Navier–Stokes equation with degenerate random forcing: Finite dimensional approximation, to appear in *Comm. Pure Appl. Math.*

[EMS] W. E, J. C. Mattingly, and Ya. Sinai, Gibbsian dynamics and ergodicity for the stochastically forced Navier–Stokes equation, to appear in *Comm. Math. Phys.*

[FM95] F. Flandoli and B. Maslowski, Ergodicity of the 2-D Navier–Stokes equation under random perturbations, *Comm. Math. Phys.* **172**:119–141 (1995).

[G] J. Goodman, Stability of the Kuramoto–Sivashinsky and related systems, *Comm. Pure Appl. Math.* **47**:293–306 (1994).

[IN] K. Ito and M. Nisio, On stationary solutions of a stochastic differential equation, *J. Math. Kyoto Univ.* **4** (1964).

[KS91] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd ed. (Springer-Verlag, 1991).

[KS1] S. Kuksin and A. Shirikyan, Stochastic dissipative PDE's and Gibbs measures, *Comm. Math. Phys.* **213**:291–330 (2000).

[KS2] S. B. Kuksin and A. Shirikyan, *On Dissipative Systems Perturbed by Bounded Random Kick-forces*, preprint (2000).

[MY] N. Masmoudi and Lai-Sang Young, *Ergodicity Theory of Infinite Dimensional Systems with Application to Dissipative Parabolic PDEs*, preprint.

[Mat98] J. C. Mattingly, *The Stochastically Forced Navier–Stokes Equations: Energy Estimates and Phase Space Contraction*, Ph.D. thesis (Princeton University, 1998).

[Mat99] J. C. Mattingly, Ergodicity of 2D Navier–Stokes equations with random forcing and large viscosity, *Comm. Math. Phys.* **206**:273–288 (1999).

[Mat00] J. C. Mattingly, Private communication, 2001.

[OKS91] B. Oksendal, *Stochastic Differential Equations*, 5th ed. (Springer-Verlag, 1991).

[T97] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics* (Springer-Verlag, New York, 1997).